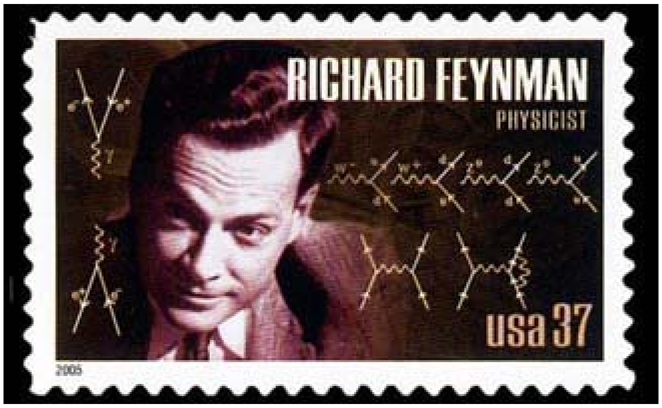
Algebra and Number Systems:

A Stunning Connection

by James D. Nickel



Source: USPS.

This essay is an attempt to exposit and amplify an erudite piece of mathematical writing by the late Richard P. Feynman (1918-1988) in his extraordinarily wonderful book *The Feynman Lectures on Physics*.[1] In chapter 22 of volume 1, he attempts to unfold the grand map of algebra. He does so as preparatory background to his subsequent study of the physics of oscillatory systems.

Feynman's gift was in teaching and seeing connections. He had an uncanny ability to explain the complex. To paraphrase him, he once said, "Unless we can explain a topic simply, then we really do not understand it."

For years I have observed with increasing dismay high school advanced algebra textbooks cover topics like number systems and logarithms (to the base 10 and to the base *e*). The reason for my dismay is that very rarely have I seen these topics presented in the context of the big picture and the principle of interconnectedness. I have had to resort to other sources, and there are many, that explain these beautifully intertwined branches, hanging resplendently with a manifold array of mathematical flora.[2] Feynman, in ten short pages, exposits the connection between number systems, algebra, logarithms, geometry, and trigonometry, in rigorously beautiful simplicity. *Every high school student of mathematics should be exposed to this type of analysis and it is to this end that I exegete and augment Feynman's gift of logical exposition.*

Feynman begins with the set of counting or natural numbers (also called positive integers). We label this set $\mathbb{N}$ or $+\mathbb{Z}$.[3] He assumes the existence of this set along with the existence of zero. The positive integers, along with zero, are sometimes called the set of whole numbers.[4] We label this set W. Hence, using set theory symbols, $W = 0 \cup \mathbb{N}$ or $W = 0 \cup +\mathbb{Z}$.[5] From this starting point, Feynman defines addition, multiplication, and exponentiation as follows:

1. Addition in $+\mathbb{Z}$ or $\mathbb{N}$: Let $a, b \in +\mathbb{Z}$ or $\mathbb{N}$ (i.e.., *a* and *b* represent numbers that are members of the set of positive integers or natural numbers). If we start with *a* and count successively one unit *b* times, the number resulting from this counting procedure is $a + b$. In other words, *addition is counting forward.*

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[1] Richard P. Feynman, *The Feynman Lectures on Physics: Commemorative Issue* (Cal Tech, [1963] 1989), I:22-1 to 22-10.

[2] The books by mathematics professor Eli Maor – *Trigonometric Delights, To Infinity and Beyond, e: The Story of a Number, The Pythagorean Theorem* – are one example of this type of exposition. Math students who read books like these will not only be intellectually challenged, but will also be drawn into what I can a "zone of mathematical beauty and elegance."

[3] $\mathbb{Z}$ comes from the German word *zahl* meaning "number." $+\mathbb{Z}$, in symbols, means the "set of positive integers."

[4] Integer, in Latin, means "whole or undivided."

[5] In set theory, the symbol $\cup$ means "union."

2. Multiplication in $+\mathbb{Z}$ or $\mathbb{N}$: Let $a, b \in +\mathbb{Z}$. If we start with zero and add a to it, b times in succession, the number resulting from this counting procedure is $b \times a$. *Multiplication is repeated addition by the same number* or, in symbols:

$$b \times a = \underbrace{a + a + \cdots + a}_{\text{b times}}$$

3. Exponentiation in $+\mathbb{Z}$ or $\mathbb{N}$: Let $a, b \in +\mathbb{Z}$. If we start with 1 and multiply by a, b times in succession, the number resulting from this counting procedure is a^b. *Exponentiation is repeated multiplication by the same number* or, in symbols:

$$a^b = \underbrace{a \times a \times \cdots \times a}_{\text{b times}}$$

Earlier in his book, Feynman seeks to answer the question, "What is gravity?" He states, "All we have done is to describe *how* the earth moves around the sun, but we have not said *what makes it go.* Newton made no hypotheses about this; he was satisfied to find *what* it did without getting into the machinery of it. *No one has since given any machinery.*"[6] A few sentences later he states, "Why can we use mathematics to describe nature without a mechanism behind it? No one knows. We have to keep going because we find out more that way."[7] It should not come as a surprise that Feynman, as a covenant *breaker*, could not account for why mathematics works. To him, mathematics "works" so let's use it (an appeal to pragmatism). Regarding the operation of counting, Feynman says, "We suppose that we already know what integers are, what zero is, and what it means to increase a number by one unit."[8] *He makes no attempt to justify why we can count* and assumes that this "accounting for counting" cannot be done. As a covenant *keeper*, professor Vern Poythress demonstrates that the Triune God of Scripture is the sure foundation for counting:

> It may surprise the reader to learn that not everyone agrees that '2 + 2 = 4' is true. But, on second thought, it must be apparent that no radical monist can remain satisfied with '2 + 2 = 4.' If with Parmenides one thinks that all is one, if with Vedantic Hinduism he thinks that all plurality is illusion, '2 + 2 = 4' is an illusory statement. On the ultimate level of being, 1 + 1 = 1. What does this imply? Even the simplest arithmetical truths can be sustained only in a world-view which acknowledges an ultimate metaphysical plurality of the world – whether Trinitarian, Polytheistic, or chance-produced plurality. At the same time, the simplest arithmetical truths also presuppose ultimate metaphysical unity for the world – at least sufficient unity to guard the continued existence of "sames." Two apples remain apples while I am counting them; the symbol '2' is in some sense the same symbol at different times, standing for the same number. So, at the very beginning of arithmetic, we are already plunged into the metaphysical problem of unity and plurality, of the one and the many. As Van Til and Rushdoony have pointed out, this problem finds its solution only in the doctrine of the ontological Trinity. For the moment, we shall not dwell on the thorny metaphysical arguments, but note only that without some real unity and plurality, '2 + 2 = 4' falls into limbo.[9]

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[6] Feynman, I:7-9.

[7] *Ibid.*

[8] *Ibid.,* I:22-1.

[9] Vern Poythress, "A Biblical View of Mathematics," *The Foundations of Christian Scholarship* (Vallecito: Ross House Books, 1976), p. 161.

Feynman was gifted by God to see connections in the physical realm *but he was blind to the ultimate connection*. The coherence between mathematics and the physical world exists *only* because of the nature of the Creator God. Mathematics works *only* because God has created the human mind to think mathematically while, at the same time, the physical creation reflects the covenantal order of the Creator in such a way that it can be modeled by mathematical propositions. Feynman blinded himself to this truth but in his daily work as a physicist he had to assume, albeit unconsciously, the truth of the Christian system every time he discovered and articulated the wondrous connections hidden in both mathematical propositions and physical reality.

With this starting point established, let's return to the structure of algebra. Feynman next lists eleven logical consequences of addition, multiplication, and exponentiation (to simplify our subsequent discussions, we shall name these eleven properties as the *Consequences*). If $a, b, c \in +\mathbb{Z}$ or $\mathbb{N}$, then:

1. $a + b = b + a$ (commutative property of addition)
2. $a + (b + c) = (a + b) + c$ (association property of addition)
3. There exists a number 0 such that $0 + a = a$ (identity element of addition)[10]
4. $ab = ba$ (commutative property of multiplication)
5. $a(bc) = (ab)c$ (association property of multiplication)
6. $1a = a$ (identity element of multiplication)
7. $a(b + c) = ab + ac$ (distributive property of multiplication over addition)

The final four properties are logical consequences of exponentiation:

8. $a^1 = a$
9. $a^b a^c = a^{b+c}$ (we shall be employing this consequence many times in the analysis that follows)
10. $(ab)^c = a^c b^c$
11. $(a^b)^c = a^{bc}$

Note that 0 and 1 have special properties. They are the identity elements of addition and multiplication respectively. These eleven properties justify almost every operation in algebra.

Next, Feynman defines the *inverse operations* of addition, multiplication, and exponentiation. Anyone proficient in algebra knows how important these operations are in solving equations. As an elementary example, we want to solve the following equation for x:

$3x + 8 = 23$

Rhetorically, this equation means "3 times a certain number plus 8 is 23."

To solve for x, we first *subtract* 8 from both members of the equation (subtraction is the inverse of addition; the "+ 8" in the equation). We get:

$3x = 15$

Next, we divide both members of this new equation by 3 (division is the inverse of multiplication; the "3 times x" in the equation). We get:

$x = 5$ (our solution)

To define the inverse operations, we start with three equations and the numbers *a, b*, and *c* that satisfy them:

Equation 1: $a + b = c$

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[10] Note, $0 \notin +\mathbb{Z}$ but $0 \in W$.

Equation 2: $ab = c$
Equation 3: $b^a = c$ (*b* is called the base and *a* is called the exponent)

We want to solve each of these equations for *b*. From Equation 1, since $a + b = c$, then $b = c – a$. This is the definition of *subtraction*. We say, "*a* subtracted from *c* is *b*." From Equation 2, since $ab = c$, then $b = \frac{c}{a}$. This is the definition of *division*. We say, "*c* divided by *a* is *b*." From Equation 3, since $b^a = c$, the $b = \sqrt[a]{c}$. This is the definition of *extraction of roots*. We say, "the a^{th} root of *c* is *b*." For example, if $2^4 = 16$, then $2 = \sqrt[4]{16}$ or "2 is the fourth root of 16."

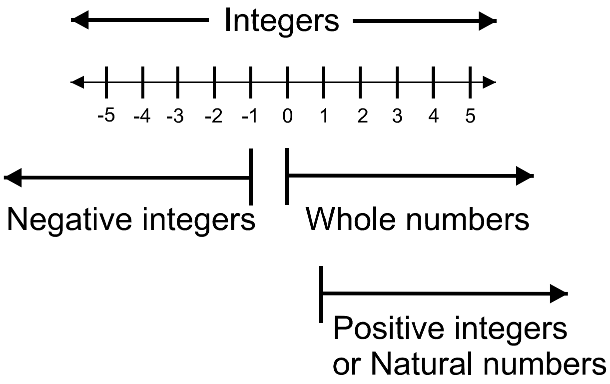
Note that $a + b = b + a$ and $ab = ba$ (commutative property). Does $b^a = a^b$? For example, does $2^3 = 3^2$? Since $2^3 = 8$ and $3^2 = 9$, then we can reasonably conjecture that there is *another* inverse of exponentiation. Given $a^b = c$, we now want to solve this equation for b. We ask, "*a* raised to what power equals *c*?" When our unknown is an exponent, we are dealing with technicalities of *logarithms*.[11] Logarithms are defined as follows: If $a^b = c$, then $b = \log_a c$. Computing logarithms and the extraction of roots are two kinds of solutions to the same type of algebraic equation (dealing with exponents). Inverse operations can now be summarized:

Table I

Operation		*Inverse*	
Addition	$a + b = c$	Subtraction	$b = c – a$
Multiplication	$ab = c$	Division	$b = \frac{c}{a}$
Exponentiation	$b^a = c$	Extraction of roots	$b = \sqrt[a]{c}$
Exponentiation	$a^b = c$	Computing logarithms	$b = \log_a c$

To this point, we have been only concerned with the properties of operations and their inverses as they apply to the positive integers ($a, b, c \in \mathbb{N}$ or $+\mathbb{Z}$). It is the inverse operations require us to both *extend* our notion of number and to *generalize* the *Consequences*.

Extension #1



In the operation of subtraction ($b = c – a$), we can let *c* and *a* be any positive integer. If $c > a$ (> stands for "greater than"), then *b*, the difference, will be positive ($b > 0$). If $c = a$, then $b = 0$. What happens if $c < a$ (< stands for "less than")? For example, compute *b* if $c = 8$ and $a = 11$; i.e., $b = 8 – 11$. Or, in terms of addition, $11 + b = 8$. What number, when added to 11, equals 8? There is no such number if we confine ourselves to the set of natural numbers or positive integers. The operation of subtraction requires us to extend the set of positive integers to include, not only 0, but the *negative integers* (If $c < a$,

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[11] Logarithm literally means "the study of number" (*logos* + *arithmos*).

then b < 0). In our example, -3 = 8 – 11. The set of integers, $\mathbb{Z}$, consists of the negative integers ($-\mathbb{Z}$), 0, and the positive integers ($+\mathbb{Z}$) or $\mathbb{Z} = -\mathbb{Z} \cup 0 \cup +\mathbb{Z}$.

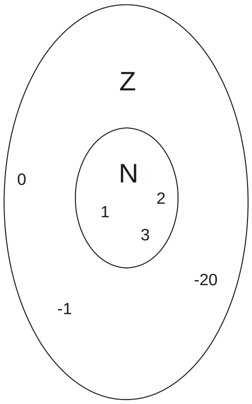
Integers allow use to solve equations like $x + 5 = -3$. Subtracting 5 from both members of these equation, we get $x = -8$.

We can also *generalize* the *Consequences* in that they hold true for $\mathbb{Z}$ and we can also use the *Consequences* to form the rules for adding, subtraction, multiplying, and dividing any integer (whether both positive, both negative, or a combination of positive and negative).

By extending our number system to include negative integers, we do run into some conceptual issues. For example, we stated that *multiplication is repeated addition by the same number* or, in symbols:

$$b \times a = \underbrace{a + a + \cdots + a}_{\text{b times}}$$

(-2)×3 makes no conceptual sense with this definition. How can you multiply 3 by itself "negative 2" times? Even though we experience conceptual failure, we can work around this and the rules still unfold. We can establish that (-2)×3 = -6.

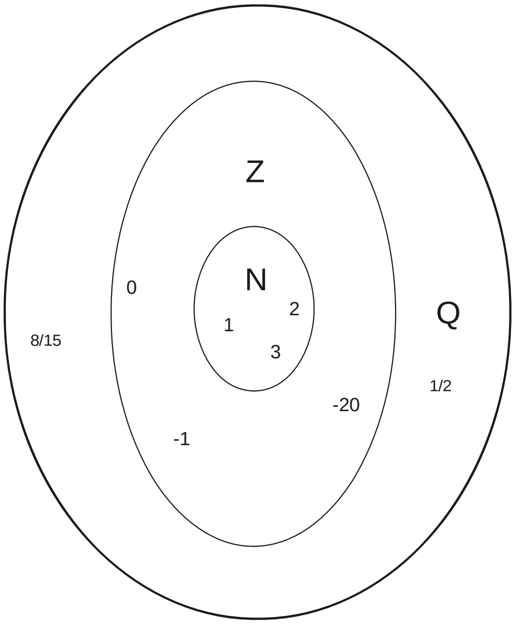


Extension #1

Extension #2

In the operation of division ($b = \frac{c}{a}$ and *b* is the quotient), we can let *c* and *a* be any integer $\mathbb{Z}$. This operation works nicely if *c* is divisible by *a*. If not, we encounter remainders. Remainders require us to extend $\mathbb{Z}$ to include *fractions*.[12] For example, if c = 1 and a = 3, then $b = \frac{1}{3}$. If c = -8 and a = 5, then $b = \frac{-8}{5} = -1\frac{3}{5}$.

The set of fractions or *rational numbers* "fill out" the proverbial number line (technically, they make the number line *everywhere dense*).[13] Rational numbers are *ratio* numbers. We define them as follows: A rational number is any number that can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. The symbol for the set of rational numbers is $\mathbb{Q}$.[14] Any number in



Extension #2

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[12] Fraction literally means "to break."

[13] By everywhere dense, we mean that between any two rational numbers, you can always find another rational number. You can compute this number by computing the average of the two given rational numbers.

$\mathbb{Z}$ can be written in a rational number "dress." For example, $2 = \frac{2}{1}$ and $-5 = \frac{-10}{2}$.

Rational numbers allow use to solve equations like $3x + 5 = -3$. Subtracting 5 from both members of these equation, we get $3x = -8$. Dividing both members of this equation by 3, we get $x = -\frac{8}{3} = -2\frac{2}{3}$.

Exponentiation and its two inverses engender some intriguing consequences and Feynman carefully exegete these wonders. Given b^a, what happens when *a* is negative? For example, let's consider b^{3-8}. What does this mean? We know that $3 - 8 = -5$ and we know this:

$$(3 - 8) + 8 = 3$$

From this, we get $b^{3-8}b^8 = b^3$ from our ninth consequence. Therefore, by our definition of division, $b^{3-8} = \frac{b^3}{b^8}$. Since $b^3 = b\times b\times b$ and $b^8 = b\times b\times b\times b\times b\times b\times b\times b$, then $b^{3-8} = \frac{\cancel{b\times b\times b}}{b\times b\times b\times b\times b\times \cancel{b\times b\times b}} = \frac{1}{b^5}$. Since $3 - 8 = -5$, then $b^{3-8} = b^{-5} = \frac{1}{b^5}$. Hence, negative exponents are reciprocals of positive exponents. In general, $b^{-a} = \frac{1}{b^a}$ and $b^a = \frac{1}{b^{-a}}$.

Let's consider $\frac{1}{b^a}$ where $b \in \mathbb{Z}$ and $a \in +\mathbb{Z}$. If *a* is an even positive integer (2, 4, 6, 8, etc.), then b^a will always be positive and $\frac{1}{b^a}$ will be a rational number $\mathbb{Q}$. For example, $\frac{1}{2^2} = \frac{1}{4}$ and $\frac{1}{(-2)^2} = \frac{1}{4}$. If *a* is an odd positive integer (1, 3, 5, 7, etc.), then b^a will be positive if *b* is positive and b^a will be negative if *b* is negative. In both cases, $\frac{1}{b^a}$ will again be a rational number $\mathbb{Q}$. For example, $\frac{1}{2^3} = \frac{1}{8}$ and $\frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$.

Next, let's consider exponents that are rational numbers. For example, let's consider $b^{\frac{3}{8}}$. By our definition of division, we know:

$$\left(\frac{3}{8}\right)\times 8 = 3$$

From our eleventh consequence, we get: $\left(b^{\frac{3}{8}}\right)^8 = b^{\left(\frac{3}{8}\right)(8)} = b^3$. Also, by our definition of extraction of roots, we get this relationship: $b^{\frac{3}{8}} = \sqrt[8]{b}^{\,3}$. With more demonstrations like these, we can conclude that the Consequences will hold true for $\mathbb{Q}$.

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[14] The letter $\mathbb{Q}$ stands for "quotient."

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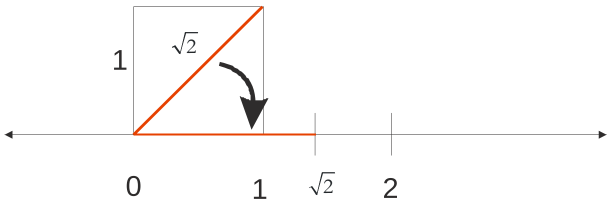
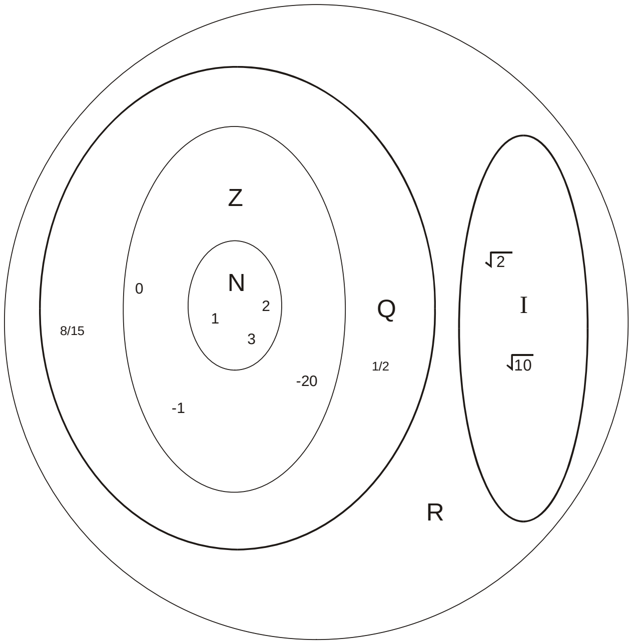
"Extension #3"

We have already discovered that both subtraction and division requires us to extend our concept of number from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$. Can we go further? Consider the equation $x^2 - 2 = 0$. What value or values of x will make this equation true? Adding 2 (inverse of subtraction) to both members of this equation, we get $x^2 = 2$. Extracting the square root of both members of this equation, we get $x = \sqrt{2}$ (we are only considering the positive root of 2). The ancient Greeks encountered this number when they considered the length of the diagonal of a unit square. By the Pythagorean Theorem, this length, when the sides of the square were 1 unit, is $\sqrt{2}$. In a stunning display of the power of *reductio ad absurdum* (indirect proof), Greek mathematicians showed that $\sqrt{2}$ could *not* be written as the ratio of two integers. Hence, $\sqrt{2}$ is *not* a rational number. It is a different kind of number. The Greeks denoted this number as *alogos* (without ratio). Today, we denote $\sqrt{2}$ as an *irrational* number. Remember when we stated that the number line is *everywhere dense* with rational numbers? $\sqrt{2}$, being a positive length, can be represented on the number line. Hence, although the number line is *everywhere dense* with rational numbers, there are gaps. Both Richard Dedekind (1831-1916) and Georg Cantor (1845-1918), German mathematicians, showed that there are an *infinite number of gaps* in the number line, gaps that are filled with irrational numbers. $\mathbb{Z}$ extends $\mathbb{N}$ (i.e., $\mathbb{N} \subset \mathbb{Z}$ or $\mathbb{N}$ is contained in $\mathbb{Z}$)[15] and $\mathbb{Q}$ extends $\mathbb{Z}$ (i.e., $\mathbb{Z} \subset \mathbb{Q}$ or $\mathbb{Z}$ is contained in $\mathbb{Q}$). The set of irrational numbers, denoted as I, does *not* extend $\mathbb{Q}$ (i.e., $\mathbb{Q} \not\subset \mathrm{I}$ or $\mathbb{Q}$ is *not* contained in I). $\mathbb{Q}$ and I are disjoint sets. Together, $\mathbb{Q}$ and I compose the set of real numbers, denoted as $\mathbb{R}$ (or $\mathbb{Q} \cup \mathrm{I} = \mathbb{R}$).





"Extension #3"

Every rational number can be written in decimal form were the decimal expansion either repeats or terminates. For example, $\frac{1}{3} = 0.\overline{3}$ where 3, the repetend, repeats *ad infinitum*. $\frac{1}{4} = 0.25$ where the decimal expansion terminates at 5 (in the hundredths

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[15] The symbol $\subset$, in set theoretical notation, means "is contained in."

position). The decimal expansion of irrational numbers like $\sqrt{2}$ neither repeats nor terminates. Because of this intriguing property, every irrational number can be approximated by a rational number to any degree of precision required.

Feynman next considered irrational exponents. Consider the equation $x = 10^{\sqrt{2}}$. This number can be approximated by estimating $\sqrt{2}$ by a rational number. For example, we let $\sqrt{2} \approx 1.4 = 1\frac{4}{10} = \frac{14}{10}$. Therefore, $10^{\sqrt{2}} \approx 10^{\frac{14}{10}} = \sqrt[10]{10^{14}}$. A better approximation can be obtained by letting $\sqrt{2} \approx 1.41 = 1\frac{41}{100} = \frac{141}{100}$. Therefore, $10^{\sqrt{2}} \approx 10^{\frac{141}{100}} = \sqrt[100]{10^{141}}$. We can get better and better estimations but note that by doing so we will be calculating very large roots (e.g., 1,000th root of 10, 10,000th root of 10, 100,000th root of 10, etc.). We get:

$$10^{\sqrt{2}} \approx 10^{\frac{1414}{1000}} = \sqrt[1000]{10^{1414}}$$

$$10^{\sqrt{2}} \approx 10^{\frac{14,142}{10,000}} = \sqrt[10,000]{10^{14,142}}$$

$$10^{\sqrt{2}} \approx 10^{\frac{141,421}{100,000}} = \sqrt[100,000]{10^{141,421}}$$

Our approximations will become significantly more difficult to compute (without the aid of calculators, of course, a tool that Feynman did not have access to in the early 1960s when his lectures on physics were first delivered at Cal Tech).[16]

Remember that there are two inverses of exponentiation. We can solve the equation $x = 10^{\sqrt{2}}$ by extraction of roots (to any degree of precision we desire) and we can solve the equation $10^x = 2$ by computing logarithms. By definition, $10^x = 2 \Leftrightarrow x = \log_{10} 2$. Hence, we just need to compute the logarithm to the base 10 of 2. How do we do this?

Feynman proceeded to explore the general "ideational mode of attack." If we can calculate 10^1, $10^{\frac{4}{10}}$, $10^{\frac{1}{100}}$, $10^{\frac{4}{1000}}$, etc. and multiply them together, we would get:

$$x = 10^1 \times 10^{\frac{4}{10}} \times 10^{\frac{1}{100}} \times 10^{\frac{4}{1000}} = 10^{1+\frac{4}{10}+\frac{1}{100}+\frac{4}{1000}} = 10^{1.414} \approx 10^{\sqrt{2}}$$

To do this, we must be able to calculate $10^{\frac{1}{10}} = \sqrt[10]{10}$, $10^{\frac{1}{100}} = \sqrt[100]{10}$, $10^{\frac{1}{1000}} = \sqrt[1000]{10}$, etc. Before the invention of calculators, these computations were tediously difficult. However, thanks to the ancient Babylonians and Isaac Newton (1642-1727), there exists a recursive algorithm whereby it is relatively easy (you have to do the computations, though) to calculate the *square root* of any number to a remarkable degree of accuracy.[17] Let's say that you want to find $\sqrt{n}$. The algorithm is based upon an initial guess, *g*. You take the average of that guess, *g*, and the quotient of $\frac{n}{g}$. Calculating this average gives you an even better approxima-

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[16] Logarithmic tables, the staple of the appendices to math and science textbooks until the late 1980s, assisted a human "computer" with these calculations.

[17] Technically, this algorithm converges very rapidly to the number sought.

tion. This approximation is then used as the next guess and the average is again taken (this is why this algorithm is recursive: its output becomes the input for the next calculation). In symbols, this algorithm works like this:

Given n, we want to compute $\sqrt{n}$. Assume $\sqrt{n} \approx g$.

Step 1. We compute a better approximation, a, where $a = \dfrac{g + \dfrac{n}{g}}{2}$.

Step 2. Let g = a.
Go to Step 1 or halt the algorithm when a reaches the precision desired.

For example, we consider computing $\sqrt{2}$. We estimate $\sqrt{2} \approx 1.4$. Hence, g = 1.4.

Step 1. $a = \dfrac{1.4 + \dfrac{2}{1.4}}{2} = 1.414286$ (rounded).
Step 2. g = a = 1.414286
We repeat:

Step 1. $a = \dfrac{1.414286 + \dfrac{2}{1.414286}}{2} = 1.414214$ (rounded). This is the actual value of $\sqrt{2}$ rounded to the nearest millionth. We can now halt this remarkably accurate process.

With this algorithm in mind, instead of calculating $10^{\frac{1}{10}} = \sqrt[10]{10}$, $10^{\frac{1}{100}} = \sqrt[100]{10}$, $10^{\frac{1}{1000}} = \sqrt[1000]{10}$, etc. we can calculate $10^{\frac{1}{2}} = \sqrt{10}$, $10^{\frac{1}{4}} = \left(10^{\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{\sqrt{10}}$, $10^{\frac{1}{8}} = \left(\left(10^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{\sqrt{\sqrt{10}}}$, etc. Before we perform these calculations, we need to ask why we are doing this work with 10 instead of another number.

As background, logarithms were invented in the 17th century to ease computations (primarily multiplication and division) of large numbers.[18] From our ninth consequence, we know that $a^b a^c = a^{b+c}$. We also know that $a^b = c \Leftrightarrow b = \log_a c$. What happens when we take the logarithm of the product of two numbers?

We let $a^b = x$ and $a^c = y$ and we want to find $\log_a(xy)$. We reason as follows:
Since $a^b = x$, then, by definition, $b = \log_a x$
Since $a^c = y$, then, by definition, $c = \log_a y$
Since $xy = a^b a^c = a^{b+c}$, then $a^{b+c} = xy$
Hence, by definition, $b + c = \log_a(xy)$
Since $b = \log_a x$ and $c = \log_a y$, then, by substitution, $\log_a(xy) = \log_a x + \log_z y$

What we have demonstrated is that a *multiplication* problem can be translated, by use of logarithms, into an *addition* problem.[19] This relationship, this law-order, holds for *any* base *a*.

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[18] 17th century problems in astronomy generated "big number" type problems.
[19] Likewise, a *division* problem can be translated, by use of logarithms, into a *subtraction* problem.

The question now focuses on the choice of a base. Let's say that we are able to determine the logarithms for a given base *a*. This means that we can solve the equation $a^b = c$ for any *c*; i.e., we can compute $\log_a c = b$ for all values of *c*. Let's say that we want to calculate the logarithm of *c* to another base, base x. We need to solve $x^{b'} = c$ or compute $\log_x c = b'$ (note, because of the different base, $b' \neq b$). Since $b' \neq b$, then b' must be a factor of b. Let's let that factor be t. Hence, $tb' = b$. Since $tb' = b$, then $t = \frac{b}{b'}$ or $b' = \frac{b}{t}$. Now we let $x = a^t$. Since we know *a* and *x*, then we can find *t*. Since $x = a^t$, then $\log_a x = t$. Next, note that $(a^t)^{b'} = a^{tb'} = a^b = c$. Hence, $\log_a c = tb'$ and $\log_x c = b' = \frac{b}{t}$. This means that the logarithm of any number *c* to the base *x* is equal to $\frac{b}{t}$ or $\frac{1}{t}$, a constant, multiplied by $b = \log_a c$. Therefore any logarithmic table, in base *a*, is equivalent to any other logarithmic table, in base x, if we multiply each logarithm by a constant. That constant is $\frac{1}{t} = \frac{1}{\log_a x}$. This analysis allows us to choose any particular base and then we can easily translate the logarithms so calculated into another base.

For convenience and by historical precedence, we start with base 10, the base of the decimal number system. Starting from base 10, as the English mathematician Henry Briggs (1561-1630) originally did, we can calculate the logarithms of any number as long as we can calculate square roots. As a result of these calculations, we shall encounter another base that will make things more elegant and become the "base" for a multitude of stunning mathematical connections.

We can compute logarithms by computing, using the Babylonian algorithm, successive square roots of 10: $10^{\frac{1}{2}} = \sqrt{10}$, $10^{\frac{1}{4}} = \left(10^{\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{\sqrt{10}}$, $10^{\frac{1}{8}} = \left(\left(10^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{\sqrt{\sqrt{10}}}$, etc. Here is a table of our results, *ten successive square roots of 10*, calculated to the nearest ten millionths:

Table IIA	
Exponent: k	10^k
1	10.0000000
$\frac{1}{2}$	3.1622777
$\frac{1}{4}$	1.7782794
$\frac{1}{8}$	1.3335214
$\frac{1}{16}$	1.1547820
$\frac{1}{32}$	1.0746078

$\frac{1}{64}$	1.0366329
$\frac{1}{128}$	1.0181517
$\frac{1}{256}$	1.0090350
$\frac{1}{512}$	1.0045073
$\frac{1}{1024}$	1.0022511

What can we conclude from these calculations? Since we know that $10^{\frac{1}{2}} \approx 3.1622777$, then $\log_{10}3.1622777 = 0.5$ (this answer is very accurate). We also know that since $10^{\frac{1}{4}} \approx 1.7782794$, then $\log_{10}1.7782794 = 0.25$.

Can we use this table to find $10^{\frac{3}{4}}$? First, we note: $10^{\frac{3}{4}} = 10^{\left(\frac{1}{2}+\frac{1}{4}\right)} = \left(10^{\frac{1}{2}}\right)\left(10^{\frac{1}{4}}\right)$ (our ninth consequence again). Second, since we want to find $10^{\frac{3}{4}}$, we let $10^{\left(\frac{1}{2}+\frac{1}{4}\right)} = x$. Since $10^{\left(\frac{1}{2}+\frac{1}{4}\right)} = x$, then $\log_{10} x = \left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{4}$. We have already established that $\log_a(xy) = \log_a x + \log_z y$. Since $\log_{10}3.162277 = 0.5$ and $\log_{10}1.7782794 = 0.25$, then $\log_{10}(3.162277 \times 1.7782794) = 0.5 + 0.25 = 0.75 = \frac{3}{4}$. Since $\log_{10}(3.1622777 \times 1.7782794) = \frac{3}{4}$, then $10^{\frac{3}{4}} = 3.1622777 \times 1.7782794 = 5.6234133$.

Based upon this example, if we can get enough numbers in column 1 to make up almost any number, then, by multiplying the proper numbers in column 2, we can compute 10^a for any a. If we keep extending the table; i.e., find 10^k when $k = \frac{1}{2048}, \frac{1}{4096}, \frac{1}{8192}$, etc., we should first note something. 10^k for a very small k generates a number slightly larger than 1. We get 1 plus a very small amount (let's denote this amount using the Greek letter *delta*, Δ). Take some time to study Table IIA and see if you can discover something. It looks like each decimal part in column 2, as k gets very small, is very close to half the preceding decimal number part. For example, rounding, note that $\frac{0.036}{2} = 0.018$, $\frac{0.018}{2} = 0.009$, $\frac{0.0090}{2} = 0.0045$, and $\frac{0.00450}{2} = 0.00225$. Hence, $10^{\frac{1}{2048}} \approx 1 + \frac{0.0022511}{2} = 1.00112555$. Hence, instead of actually calculating these square roots, *we can estimate them*. Furthermore, we can guess the *ultimate limit* or threshold of these roots. In other words, if we compute $\frac{\Delta}{1024}$ and let Δ get very, very small ($\Delta \to 0$), what will be the answer? It will be a number very close to 0.0022511Δ *but not exactly*. We can get a better value of this number by performing trick in calculation. We subtract the 1 from 10^k and then divide by k. In essence, we calculate $\frac{10^k-1}{k}$. Let's do that now and add four columns to Table IIA to generate Table IIB:

Table IIB

Exponent: k	1024k	10^k	$\Delta\left(10^k\right)$ (4 places, decimal part)	$\frac{10^k-1}{k}$	$\Delta\left(\frac{10^k-1}{k}\right)$ (4 places, decimal part)
1	1024	10.0000000		9.0000	
$\frac{1}{2}$	512	3.1622777		4.3246	
$\frac{1}{4}$	256	1.7782794		3.1131	
$\frac{1}{8}$	128	1.3335214		2.6682	
$\frac{1}{16}$	64	1.1547820	1787	2.4765	1917
$\frac{1}{32}$	32	1.0746078	802	2.3874	891
$\frac{1}{64}$	16	1.0366329	380	2.3445	429
$\frac{1}{128}$	8	1.0181517	184	2.3234	211
$\frac{1}{256}$	4	1.0090350	91	2.3130	104
$\frac{1}{512}$	2	1.0045073	45	2.3077	53
$\frac{1}{1024}$	1	1.0022511	23	2.3051	26
				↓	26
$\frac{\Delta}{1024}$ as $\Delta \to 0$		1 + 0.0022486Δ (correct limiting value)	←	2.3025 (correct limiting value is 2.3026)	

Note that with our calculation trick, column 6 differences are very close to column 4 differences. The "division by 2" scenario with the decimal part difference also holds. In column 6, to four decimal places, we note that we will eventually have a constant difference of 26. Why? As we keep extended the rows in this table, the differences become approximately 13, 7, 3, 2, and 1 or 13 + 7 + 3 + 2 + 1 = 26 and we get 2.3025 as our actual value (the limiting value, to four decimal places, of $\frac{10^k - 1}{k}$ as $k \to 0$ is actually 2.3026). Using technical symbols, $\lim_{k\to 0}\left(\frac{10^k - 1}{k}\right) \approx 2.3026$. Hence, the difference is really 25. We subtract 0.0000025 from the decimal part of 1.0022511 and we get 0.002251 – 0.0000025 = 0.0022486. Hence, our limiting value for our third column, 10^k, is 1 + 0.0022486Δ. Using technical terms, $\lim_{\Delta\to 0}\left(10^{\frac{\Delta}{1024}}\right) \approx 1 + 0.0022486\Delta$. For example, let $\Delta = \frac{1}{1,000,000}$. Then, $10^{\frac{\frac{1}{1,000,000}}{1024}} = 1 + 0.0022486\left(\frac{1}{1,000,000}\right) = 1.000000002$. This answer can also be calculated using the formula $1 + 2.3026\left(\frac{\Delta}{1024}\right)$ because $\frac{2.3026}{1024} = 0.0022486$.

From this analysis, let's actually calculate a logarithm. We are going to follow the reasoning and calculation process used by Briggs over three centuries ago (1620). Our task is to calculate the logarithm of 2 or to find x in the equation $\log_{10}2 = x$. Or, we find x such that $10^x = 2$. We know that $\log_{10}1.7782794 = 0.25$ and $\log_{10}3.1622777 = 0.5$. Hence, $\frac{1}{4} < \log_{10}2 < \frac{1}{2}$. This gives us a window to work toward.

Since $\log_{10}1.7782794 = 0.25$, then $10^{\frac{1}{4}} = 1.7782794$. We know that $10^{\frac{1}{4}}$, being less than 2, *will be a factor of 2*. Remember, $10^x = 2$. We proceed to factor $10^{\frac{1}{4}}$ from 2 as follows[20]:

$$\frac{10^x}{10^{\frac{1}{4}}} = 10^{x-\frac{1}{4}} = \frac{2}{1.7782794} = 1.124682$$

This number, 1.124682, is now the number whose logarithm we must find. We look to the table to find the next number that is *less than* 1.124682 … it is 1.0746078 = $10^{\frac{1}{32}}$. 1.0746078 is another factor of 2. We proceed to factor this number, $10^{\frac{1}{32}}$, from 1.124682 as follows:

$$\frac{10^{x-\frac{1}{4}}}{10^{\frac{1}{32}}} = 10^{x-\left(\frac{1}{4}+\frac{1}{32}\right)} = \frac{1.124682}{1.0746078} = 1.046598$$

This number, 1.046598, is now the number whose logarithm we must find. We again look to the table to find the next number that is *less than* 1.046598 … it is 1.0366329 = $10^{\frac{1}{64}}$. 1.0366329 is another factor of 2. As before, we factor this number, $10^{\frac{1}{64}}$, from 1.046598 as follows:

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[20] We can also derive and implement this law involving exponents: $\frac{a^b}{a^c} = a^{b-c}$.

$$\frac{10^{x-\left(\frac{1}{4}+\frac{1}{32}\right)}}{10^{\frac{1}{64}}}=10^{x-\left(\frac{1}{4}+\frac{1}{32}+\frac{1}{64}\right)}=\frac{1.046598}{1.0366329}=1.009613\,.$$

We continue this recursive algorithm. This number, 1.009613, is now the number whose logarithm we must find. We again look to the table to find the next number that is *less than* 1.009613 … it is 1.0090350 $=10^{\frac{1}{256}}$. We factor this number, $10^{\frac{1}{256}}$, from 1.009613 as follows:

$$\frac{10^{x-\left(\frac{1}{4}+\frac{1}{32}+\frac{1}{64}\right)}}{10^{\frac{1}{256}}}=10^{x-\left(\frac{1}{4}+\frac{1}{32}+\frac{1}{64}+\frac{1}{256}\right)}=\frac{1.009613}{1.0090350}=1.000573\,.$$

This number, 1.000573, is now the number whose logarithm we must find. We look to the table to find the next number that is *less than* 1.000573 … but this number is *beyond* the calculated limits of our table, i.e., $\left(10^{\frac{1}{1024}}=1.0022511\right)$. To calculate the logarithm of this factor, we use our result,

$10^{\frac{\Delta}{1024}}\approx 1+2.3026\left(\frac{\Delta}{1024}\right)$. We know that $1.000573=1+2.3026\left(\frac{\Delta}{1024}\right)$. Solving for Δ, we get:

$\Delta=\frac{(0.000573)(1024)}{2.3026}=0.255\,.$

By this recursive process, we have been able to factor 2 into the following:

$$2\approx(1.7782794)(1.0746078)(1.0366329)(1.0090350)(1.000573)$$

$$\text{or } 10^{x}=10^{x-\left(\frac{1}{4}+\frac{1}{32}+\frac{1}{64}+\frac{1}{256}+\frac{0.255}{1024}\right)}$$

$$\text{where } \frac{1}{4}+\frac{1}{32}+\frac{1}{64}+\frac{1}{256}+\frac{0.255}{1024}\approx x$$

We recall that 1.7782794 = $10^{\frac{1}{4}}$, 1.0746078 = $10^{\frac{1}{32}}$, 1.0366329 = $10^{\frac{1}{64}}$, 1.0090350 = $10^{\frac{1}{256}}$, and 1.000573 = $10^{0.255}$. Therefore, we can estimate $10^{x}=2$ as:

$$\left(10^{\frac{1}{4}}\right)\left(10^{\frac{1}{32}}\right)\left(10^{\frac{1}{64}}\right)\left(10^{\frac{1}{256}}\right)\left(10^{\frac{0.255}{1024}}\right)=10^{\frac{1}{4}+\frac{1}{32}+\frac{1}{64}+\frac{1}{256}+\frac{0.255}{1024}}=10^{0.30103}\approx 2$$

Hence, $\log_{10}2\approx 0.30103$ (this answer is *accurate* to five decimal places).

Calculating this way, it took Mr. Briggs *many, many years* working with pencil and paper (before calculators and computers) to generate the logarithmic tables that used to grace the appendices in science and math textbooks. According to Feynman, Mr. Briggs was reportedly to have said, "I computed successively 54 square roots of 10."[21] If the above calculations tired you as they did me, then hats off to Mr. Briggs! He ac-

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[21] Feynman, I:22-6.

tually calculated 27 successive square roots of 10 and used the Δ formula to calculate the other 27. He also calculated his answers to *16 decimal places*, rounding off to 14 in his published tables. Today, logarithmic tables are computed using expansions of series.[22]

Before we end this excursion in tedious computation of logarithms, we need to especially note that for small exponents *k* (or, as $k \to 0$), we can easily calculate 10^k by using the fact that $10^k = 1 + 2.3026k$. We can state this relationship in another way. We also note that $10^{\frac{n}{2.3026}} = 1 + n$ as $n \to 0$. Why? We know that $k = \frac{\Delta}{1024}$. Hence, $10^k = 1 + 2.3026\left(\frac{\Delta}{1024}\right)$. Let $n = 2.3026\left(\frac{\Delta}{1024}\right)$. Hence, $2.3026\left(\frac{\Delta}{1024}\right) = 2.3026k = n$. Hence, $k = \frac{n}{2.3026}$. By substitution, $10^{\frac{n}{2.3026}} = 1 + n$.

Before, we noted that logarithms to any other base are simply multiples of logarithms to the base 10. We chose base 10 because it is convenient (we use a base 10 decimal system) and, for this reason, Briggs starting with this base. Is there a way in which we can change the scale of our logarithms to a naturally mathematical one? Since $10^{\frac{n}{2.3026}} = 1 + n$, then we can proceed to multiply all the logarithms to the base 10 by 2.3026. Our answers will correspond to another base, our mathematically *natural* base. Let this base be signified by the letter *e*. Note that $10^{\frac{n}{2.3026}} = 1 + n \Leftrightarrow \log_{10}(1+n) = \frac{n}{2.3026}$. Multiplying both members of this equation by 2.3026, we get $(2.3026)\log_{10}(1+n) = n$. Since $(2.3026)\log_{10} = \log_e$ (our "natural" definition), then $(2.3026)\log_{10}(1+n) = \log_e(1+n) \approx n \Leftrightarrow e^n = 1 + n$ as $n \to 0$. Note that this expression, $e^n = 1 + n$ as $n \to 0$, is very clean and efficient. *It is a "natural."* Compare it with the somewhat cumbersome $10^{\frac{n}{2.3026}} = 1 + n$.

What is the value of *e* that generates this efficiency? We know that $e^n = 1 + n$ and $10^{\frac{n}{2.3026}} = 1 + n$. Hence, $e^n = 10^{\frac{n}{2.3026}}$. Letting n = 1, then $e^1 = e = 10^{\frac{1}{2.3026}}$. Hence, $10^{\frac{1}{2.3026}} \approx 10^{0.43429\ldots}$. Since $\lim_{k \to 0}\left(\frac{10^k - 1}{k}\right) \approx 2.3026$, then the exponent of 10, 0.43429 …, is *indeed* an irrational number. We can now invoke our table to approximate this irrational number. We must solve this equation for e: $10^{0.43429\ldots} = e$. Without going into the detail of the calculations (this is left to the reader to compute), we can estimate *e*, rounded to 4 decimal places, as follows:

$$e = (1.7782794)(1.3335214)(1.0746078)(1.0366329)(1.0181517)(1.0090350)(1.001643) \approx 2.7184^{23}$$

As previously hinted, *e* is a dazzling number. It is the base of the *natural* logarithms. Calculated from our table of successive square roots of 10, it is a number that ties together a host of mathematical propositions. These connections are so unbelievable that some mathematicians have denoted *e* as *miraculous*. Mathematician Eli Maor has written a 215-page book exploring some of these connections. It is entitled *e*: *The Story of a Number* (Princeton University Press, 1994). In the final four pages of his exposition, Feynman will proceed to unearth this "jewel" in a way calculated to stun and awe the beholder.

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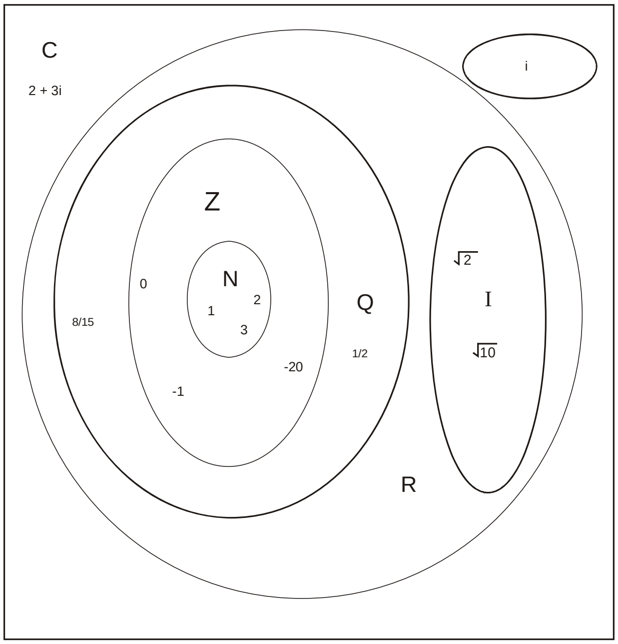
[22] A series is the sum of a patterned sequence of numbers.
[23] The actual value of e, to ten decimal places, is 2.7182818284.

Algebra and Number Systems:
A Stunning Connection

by James D. Nickel

Extension #4

Let's retrace the steps we have taken through number systems. We started with the set of natural numbers, ℕ or +ℤ. To solve simple algebraic equations, we saw the need to extend this set by adding 0 and the opposites, or negative integers, -ℤ. +ℤ, 0, and -ℤ comprise the set of integers, ℤ. Fractions, required to solve certain algebraic equations, extend ℤ to ℚ. Finally, we studied irrational numbers, I, a set of numbers disjoint from ℚ. The set of rational numbers and the set of irrational numbers, taken together or joined, generate the set of real numbers, ℝ.

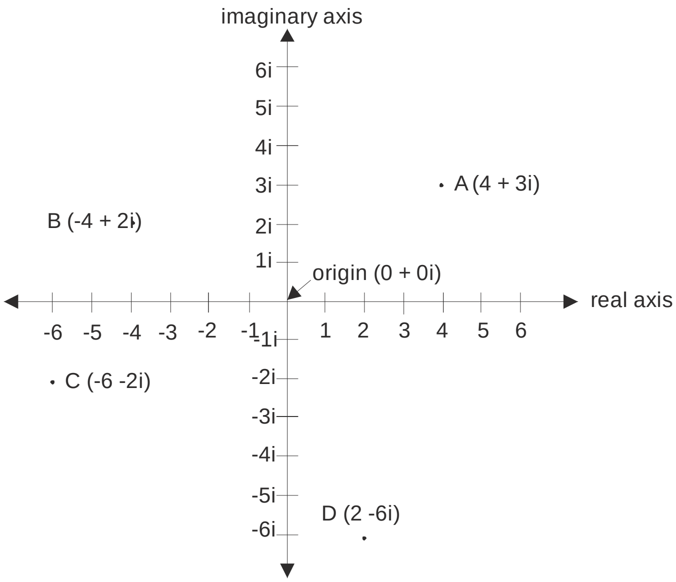


Extension #4

We have one final extension to make. We saw that irrational numbers are necessary to solve an equation like $x^2 - 2 = 0$. Consider the seemingly harmless equation $x^2 + 1 = 0$. What value or values of x make this equation true? To solve, we subtract 1 from both members of the equation. We get $x^2 = -1$. Next, we extract the square root from both members of the equation. We get $x = \sqrt{-1}$. This "solution" leaves us in a quandary. To find $\sqrt{-1}$, we must find a number, when multiplied by itself, equals -1. There is no such number, at least in the set ℝ.

This dilemma requires us to extend ℝ to include numbers some of which, when multiplied by themselves, generate a product that is a negative number.

We start this number system by letting the letter *i* stand for a number that satisfies the condition $i^2 = -1$ or $i = \sqrt{-1}$. This letter *i* stands for the *unit or pure imaginary* number. Note that if the square of i is i^2, then the square of -i is also i^2. Why? $(-i)(-i) = i^2$. Because of this property, -i is called the *complex conjugate* of i.[24] Hence, in the imaginary realm, there are two solutions to the equation $x^2 + 1 = 0$; they are i and –i.



Men from three different countries, the German mathematician Carl Friedrich Gauss (1777-1855), the Norwegian surveyor Caspar Wessel (1745-1818), and the French amateur mathematician/book store man-

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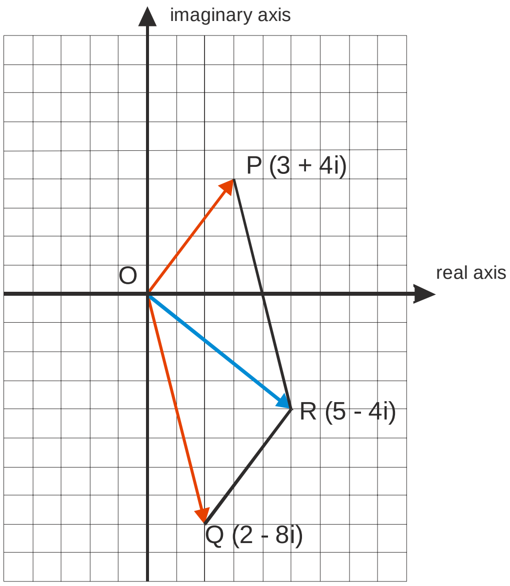
[24] Conjugate, in Latin, means "to yoke together."

ager Jean Robert Argand (1768-1822), suggested an amazing way to represent imaginary numbers. Since the time of René Descartes (1596-1650), mathematicians used coordinate systems to picture equations in two unknowns. Every high school student learns this system with its x-axis, y-axis, origin, ordered pairs, and four quadrants. Analytical geometry provides a way to "visualize" solutions to algebraic equations. Gauss used coordinate geometry as a way to "visualize" imaginary numbers. He let the x-axis (the horizontal axis) represent $\mathbb{R}$. It was simply a representation of our proverbial number line, the union of rational and irrational numbers. He then let the y-axis (the vertical axis) represent positive and negative imaginary numbers (with *i* as the imaginary unit). Hence, every point on this grid of four quadrants represents a "number" consisting of a real number part and an imaginary number part. Numbers like this, numbers in the form a + bi where $a, b \in \mathbb{R}$ are called complex numbers $\mathbb{C}$. Hence, every real number $\mathbb{R}$ is a complex number where b = 0. For example, $\sqrt{2}$ can be written in a "complex number dress," $\sqrt{2} + 0i$ where $a = \sqrt{2}$ and b = 0. Because of this designation, the set of real numbers $\mathbb{R}$ are contained in the set of complex numbers $\mathbb{C}$ ($\mathbb{R} \subset \mathbb{C}$).

There is an arithmetic to complex numbers that is fascinating. For example, adding two complex numbers is just a matter of adding their corresponding real number parts and their corresponding imaginary number parts. In general, $(a + bi) + (c + di) = (a + c) + (b + d)i$. For example, $(3 + 4i) + (2 - 8i) = (5 - 4i)$. Scientists, upon seeing the graphical representation of the addition of complex numbers, immediately understood the complex number sum as representing the resultant vector of two independent forces (based upon Isaac Newton's parallelogram law of addition of forces).



Vector Addition

Likewise, for subtraction, $(a + bi) - (c + di) = (a - c) + (b - d)i$. Multiplying two complex numbers works out like this:

$(a + bi)(c + di) = ac + adi + bci + bdi^2 =$
$ac + i(ad + bc) + bdi^2$
Since, by definition, $i^2 = -1$, then $ac + i(ad + bc) + bdi^2 = ac + i(ad + bc) - bd = (ac - bd) + i(ad + bc)$

Successive powers of i are denoted in this table:

$i =$	$\sqrt{-1}$
$i^2 = (\sqrt{-1})(\sqrt{-1}) =$	-1
$i^3 = (\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = -1(i) =$	$-i$
$i^4 = (\sqrt{-1})(\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = (-1)(-1)$	$+1$

Any larger power of *i* can be reduced to one of these basic four. For example:

$i^5 = i^{4+1} = i^4 i^1 = (1)(\sqrt{-1}) = (\sqrt{-1}) = i$
$i^{15} = i^{4+4+4+3} = i^4 i^4 i^4 i^3 = (1)(1)(1)(-i) = -i$

We can now complete the table of the powers of i. Note especially the "cycling" or "periodic" nature of this table. This observation will come in handy later.

i	=	$\sqrt{-1}$
i^2	=	-1
i^3	=	-i
i^4	=	+1
i^5	=	i
i^6	=	-1
i^7	=	-i
i^8	=	+1
i^9	=	i
i^{10}	=	-1
i^{11}	=	-i
i^{12}	=	+1
i^{13}	=	i
i^{14}	=	-1
i^{15}	=	-i

Let's see what $(-3i)^2$ and $+(3i)^2$ equal:

$+(3i)^2 = (3i)(3i) = 9i^2 = 9(-1) = -9$
$(-3i)^2 = (-3i)(-3i) = 9i^2 = 9(-1) = -9$

The complex conjugate of a + bi is a – bi. The complex conjugate of bi is -bi. If we multiply a complex number by its conjugate, we get:

$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$ (the imaginary part disappears)
$(bi)(-bi) = (b)(-b)i^2 = (b)(-b)(-1) = (b)(b) = b^2$ (again, the imaginary part disappears)

In 1799, at the age of 22, Gauss showed, by a proof that is beautiful, elegant, but not at all intuitive, that with this extension of $\mathbb{R}$ to $\mathbb{C}$, *every algebraic equation can be solved.* Technically, this proof, the Fundamental Theorem of Algebra, states that a polynomial[25] of degree *n* has exactly *n* complex solutions (or roots). Also, the *Consequences* hold true for $\mathbb{C}$. With $\mathbb{C}$, there is no need to extend the number systems any further. For example, $\sqrt{i}$ is not a "new" number. i^i is not a new number. $\mathbb{C}$ is *sufficient* for the solution of every polynomial equation; i.e., $\mathbb{C}$ encapsulates everything we need to have to solve any equation written algebraically, i.e., an equation written in terms of a finite number of algebraic symbols.

Operations with complex numbers introduce us into some fascinating realms. Journeying through this dominion is like investigating the visual wonders of Carlsbad Caverns. In the final words of his exposition, Feynman crawls through a small opening in this vast cave and excavates the intriguing treasure unearthed by *computing complex powers of complex numbers.*

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[25] A general polynomial equation of degree *n* is of the form $y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where a_n is the coefficient of x^n, a_{n-1} is the coefficient of x^{n-1}, etc., down to a_0, which is the coefficient of x^0 (or 1).

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Let's start by simplifying the situation. Instead of trying to compute complex powers of complex numbers, let's compute complex powers of real numbers. We shall consider $10^{a + bi}$. By our ninth consequence, $10^{a + bi} = 10^a 10^{bi}$. We already know how to compute 10^a for any $a \in \mathbb{R}$. We also know how to multiply something by something else. So, all we need to do is figure out how to compute 10^{bi}. Since we are raising a real number to an imaginary power, we can reasonable conclude that our answer will be a complex number. We let this answer be x + yi. Hence, we get:

$$10^{bi} = x + yi$$

If this is true, then we can write an equation that is true for its respective conjugates. The conjugate of bi is -bi and the conjugate of x + yi is x – yi. We get:

$$10^{-bi} = x - yi$$

Now, we multiply 10^{bi} by 10^{-bi}. We get:

$$(10^{bi})(10^{-bi}) = 10^0 = 1 = (x + yi)(x - yi) = x^2 + y^2$$

Thus, $1 = x^2 + y^2$. This means if we can find x, then we can find y. Since $y^2 = 1 - x^2$, then $y = \sqrt{1 - x^2}$.

We now ask ourselves, "How do we compute 10 to an imaginary exponent?" "How do we compute 10^{bi} for a specific value of *b*?" Feynman guides us along narrow walls of this cave by supposing that if we can compute it for any particular *b*, then we can compute it for everything else, b^2, 2b, 3b, $\sqrt{b}$, etc. Feynman now invokes a result from our work with logarithms. We now that $10^k = 1 + 2.3026k$ as $k \to 0$. This works if $k \in \mathbb{R}$. Feynman makes a *leap of intuition* and says, in effect, "Let's assume this works for $k \in \mathbb{C}$ and let's see what happens." Hence, if $k \in \mathbb{C}$, then k = bi. Hence, $10^{bi} = 1 + 2.3026(bi)$ as $b \to 0$. By smallness, let's let *b* be a very small part of 1024.

With this preliminary work behind use, we can now compute all the imaginary powers of 10; i.e., we can compute x and y. Let's start with by letting $b = \dfrac{1}{1024}$. We get:

$$10^{\frac{i}{1024}} = 1.00000 + 2.3026i\left(\frac{1}{1024}\right)$$

or

$$10^{\frac{i}{1024}} = 1.00000 + 0.0022486i$$

Note that in our calculations we are going to limit our precision to five significant figures in the decimal part. If we multiply $10^{\frac{i}{1024}}$ by $10^{\frac{i}{1024}}$, we get: $\left(10^{\frac{i}{1024}}\right)\left(10^{\frac{i}{1024}}\right) = 10^{\frac{i}{1024} + \frac{i}{1024}} = 10^{\frac{i}{512}}$.

What is $10^{\frac{i}{512}}$ equal to? We multiply $1.00000 + 0.0022486i$ by $1.00000 + 0.0022486i$. We get:

$$\begin{aligned}&(1.00000 + 0.0022486i)(1.00000 + 0.0022486i) = \\ &1.00000 + 0.0044972i - 0.00000505 = 1.00000 + 0.00450i\end{aligned}$$

$10^{\frac{i}{512}} = 1.00000 + 0.00450i$ (to five significant figures in the decimal part). We can continue this squaring process, $\left(10^{\frac{i}{512}}\right)^2 = 10^{\frac{i}{256}}$ and, by doing so, we can generate Table III.

Table III
Successive squares of
$10^{bi} = 10^{\frac{i}{1024}} = 1.00000 + 0.0022486i$

Exponent: bi	1024b	$10^{bi} = x + yi$
$\frac{i}{1024}$	1	1.00000 + 0.00225i (rounded)
$\frac{i}{512}$	2	1.00000 + 0.00450i
$\frac{i}{256}$	4	0.99996 + 0.00900i
$\frac{i}{128}$	8	0.99984 + 0.01800i
$\frac{i}{64}$	16	0.99936 + 0.03599i
$\frac{i}{32}$	32	0.99742 + 0.07193i
$\frac{i}{16}$	64	0.98967 + 0.14349i
$\frac{i}{8}$	128	0.95885 + 0.28402i
$\frac{i}{4}$	256	0.83872 + 0.54467i
$\frac{i}{2}$	512	0.40679 + 0.91365i
$\frac{i}{1}$	1024	-0.66928 + 0.74332i

Inspect the table for a few minutes and draw some conclusions. In column 3, we have a representation of x + yi. Notice that x starts as positive and then moves to negative. What significance is this? We shall see in a moment. Note that for each x-value and y-value in column 3, $x^2 + y^2 \approx 1$. If we did not invoke rounding or if we carried our precision to more decimal places, we would discover that, indeed, $x^2 + y^2 = 1$. Hence, Feynman's intuitive leap is paying off. For what number *b* is the real number part of 10^{bi} equal to 0? The y-term would be i so we would have $10^{bi} = i \Leftrightarrow bi = \log_{10} i$. Just as we calculated $\log_{10} 2$ using Table IIB, we can calculate $\log_{10} i$ using Table III. Without going into the detail, $\log_{10} i = 0.68226i \Leftrightarrow 10^{0.68226i} = i$.[26]

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[26] You can verify that $\log_{10} i = 0.68226i$ using a scientific calculator like TI-83 Plus (by Texas Instruments).

In Table III, we squared the exponents each time. What happens if we let the exponents increase arithmetically? By doing this, we will get a closer look at what is happening with the minus signs. So, in the next table, we are going to explore what happens to $10^{\frac{i}{8}}$ as we increase the exponents arithmetically.

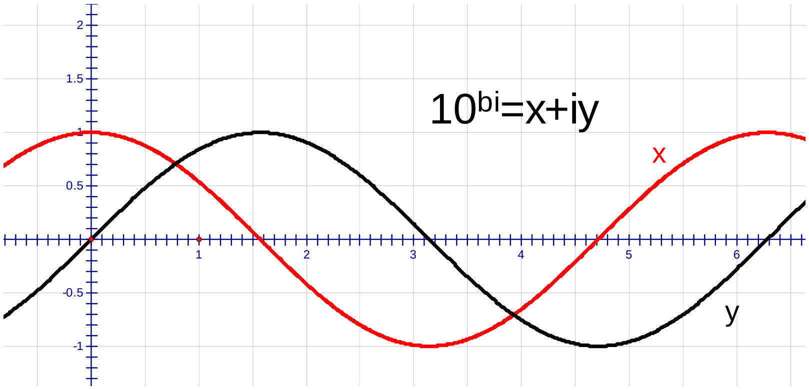
Table IV

Successive powers of $10^{\frac{i}{8}}$

m = exponent × 8i	$10^{\frac{im}{8}}$
0	1.00000 + 0.00000i
1	0.95882 + 0.28402i
2	0.83867 + 0.54465i
3	0.64944 + 0.76042i
4	0.40672 + 0.91356i
5	0.13050 + 0.99146i
6	-0.15647 + 0.98770i
7	-0.43055 + 0.90260i
8	-0.66917 + 0.74315i
9	-0.85268 + 0.52249i
10	-0.96596 + 0.25880i
11	-0.99969 – 0.02620i
12	-0.95104 – 0.30905i
14	-0.62928 – 0.77717i
16	-0.10447 – 0.99453i
18	0.45454 – 0.89098i
20	0.86648 – 0.49967i
22	0.99884 + 0.05287i
24	0.80890 + 0.58836i

When m = 0, $10^{\frac{i(0)}{8}} = 10^0 = 1$. When m = 1, $10^{\frac{i(1)}{8}} = 10^{\frac{i}{8}}$. These values were calculated in Table III. When m = 2, $10^{\frac{i(2)}{8}} = 10^{\frac{i}{4}}$. Again, these values were calculated in Table III. When m = 3, $10^{\frac{i(3)}{8}} = 10^{\frac{3i}{8}}$. We know that $\left(10^{\frac{i}{4}}\right)\left(10^{\frac{i}{8}}\right) = 10^{\frac{3i}{8}}$. Hence, $10^{\frac{3i}{8}}$ = (0.95882 + 0.28402i)(0.83867 + 0.54465i) = 0.64944 + 0.76042i. The rest of the table can be filled out using the ninth consequence and values from Table III.

What do we notice? We see that x starts from 1, decreases, passes through 0, and continues to -1. Then, x starts



increasing again, passes through 0, and marches to 1. Regarding y, y starts from 0, increases to 1, then decreases, passes through 0, continues to -1, and then increases and passes through 0. *Anyone who knows anything about trigonometry ought to be stunned by this revelation.* This is nothing but a description of the attributes of the sine function and cosine function.

Why does 10^{bi} repeat or oscillate in such a manner? We know that $10^{0.68226i} = i$. Then, $\left(10^{0.68226i}\right)^2 = 10^{1.36452i} = i^2 = -1$. Next, note that $\left(10^{0.68226i}\right)^4 = 10^{2.72904i} = (i^2)^2 = i^4 = +1$. This analysis should confirm the cyclic or periodic behavior of these powers.

Feynman's next step is a step into mathematical glory. Instead of using base 10, he translates these values into the natural base, base $e \approx 2.7182818284$. We started with 10^{bi}. As before, we let $t = 2.3025b$ (note, $t \in \mathbb{R}$) and write $10^{bi} = e^{ti}$. Since $10^{bi} = x + yi$, then $e^{ti} = x + yi$. Since x behaves like the algebraic cosine of *t* (since $t \in \mathbb{R}$) and y behaves like the algebraic sine of *t* (since $t \in \mathbb{R}$), we can then write:

$$e^{ti} = x + yi = \cos(t) + \sin(t)i$$

What are the properties of cos(t) and sin(t)? Since $x^2 + y^2 = 1$, then $\cos^2(t) + \sin^2(t) = 1$, a property that is usually established by using the Pythagorean Theorem and right triangles. This is truly a *marvelous* connection. We also know that as $t \to 0$, $e^{ti} = 1 + ti$. Hence, as $t \to 0$, $\cos(t) = 1$ and $\sin(t) = 0$. If t = degrees or radians, then, by use of right triangles and the unit circle, we can also establish that $\cos(0) = 1$, and $\sin(0) = 0$. Hence, as Feynman takes careful note, "*all of the various properties of these remarkable functions*, which come from taking imaginary powers, *are the same as the sine and cosine of trigonometry.*"[27]

What about the periodicity? Do trigonometric functions and imaginary powers cohere cyclically? To find out, we must determine x when $e^x = i \Leftrightarrow \log_e i = x$. Note, the successive powers of i (i, i^2, i^3, etc.) form the "x-axis" of our graph that pictures what happens to x and y in $e^{ti} = x + yi$. The value of x will give us the period from 0 to i. We know that $\log_{10} i = 0.68226i$. Multiplying by the scale factor, 2.3026, we get $\log_e i = 2.3026(\log_{10} i) = 2.3026(0.68226i) = 1.5710i$. On the horizontal axis, when it measures 1.5710i, then, on the vertical axis, graph will be equal to 1 for y, or sin(t), and 0 for x, or cos(t). Lo and behold, in radians, $\frac{\pi}{2} \approx 1.5708$ (remarkably close to 1.5710), and $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$. For the period from 0 to i^2, we must determine x when $e^x = i^2 \Leftrightarrow \log_e i^2 = x$. By a property of logarithms, we know that $\log_e i^2 = 2 \log_e i$. Hence, $2(1.5710i) = 3.142$. Look familiar? $\pi \approx 3.142$. Note also, $\sin(\pi) = 0$ and $\cos(\pi) = -1$. Again, these values are a perfect match with our above graph. Wonders of connection multiply!

Note carefully, using purely and only by algebra (no triangles, no unit circles), we arrived at natural logarithms and consequent values that are *natural* to geometry and trigonometry. Hence, we can replace t by θ, designating either radians or degrees, and write what Feynman pronounces as "our jewel." I might add, it is a stunning and exquisite jewel:

$$e^{\theta i} = \cos(\theta) + \sin(\theta)i$$

Feynman concludes his exposition by connecting geometry to algebra by representing a given complex number x + yi in a plane. The distance from the origin to the point that represents x + yi is r, called the

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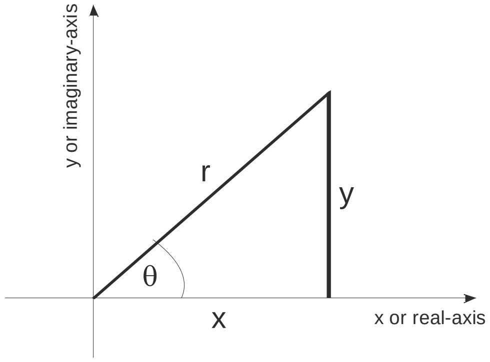
[27] Feynman, I:22-9.

modulus (meaning "measure") or magnitude of x + yi. The phase angle (a physics term), also called the *argument* or *amplitude*, of x + yi, is θ. By the Pythagorean Theorem, $r^2 = x^2 + y^2 \Leftrightarrow r = \sqrt{x^2 + y^2}$ and by trigonometry, $\tan(\theta) = \frac{y}{x}$.

Also, by trigonometry, $\cos(\theta) = \frac{x}{r} \Leftrightarrow x = r\cos(\theta)$ and $\sin(\theta) = \frac{y}{r} \Leftrightarrow y = r\sin(\theta)$. Since $e^{\theta i} = x + yi$, then, by substitution, $x + yi = e^{\theta i} = r\cos(\theta) + r\sin(\theta)i = r[\cos(\theta) + \sin(\theta)i]$. Since $e^{\theta i} = \cos(\theta) + \sin(\theta)i$ then, by substitution, $x + yi = re^{\theta i}$. This equation, $x + yi = re^{\theta i}$, is, according to Feynman, "the unification of algebra and geometry."[28]



$x + yi = re^{\theta i}$

In conclusion, note that we started this essay only with the notions of the existence of positive integers and zero. From this starting point, Feynman defined the procedure of counting. These notions and the procedure of counting, assumed by Feynman, can be justified only because of the Triune nature of God. Hence, the God of Scripture is the *Alpha* of mathematics. From counting we unfolded the basic arithmetic operations, their inverses, and the "eleven consequences." Using algebraic equations and the process of generalization, we methodically extended number systems from (ℕ or +ℤ) and 0 to ℤ. Then, we traveled from ℤ to ℚ. Next stop: I and ℝ. Final destination: ℂ. Then, we developed useful mathematical objects like tables of logarithms, powers, and trigonometric functions and discovered the remarkable connection that the sine function and cosine function *are what the imaginary powers of real numbers are*. We discovered this astonishing correlation, this *Omega* of our thinking (remember, *without Alpha there cannot be Omega*), by reasoning from the construction of a table that *merely extracted ten successive square roots of ten!*

In his next chapter of *Lectures on Physics*, entitled "Resonance," Feynman uses this "jewel" of an equation, $e^{\theta i} = \cos(\theta) + \sin(\theta)i$, to explain and make sense of the physics of harmonic motion, forced oscillation and damping, electrical resonance, and resonance in the physical creation!

The Triune God of Scripture is the Alpha and Omega of mathematics.

We are not finished with these mathematical wonders. Leonhard Euler (1707-1783), Swiss mathematician par excellence, derived the same equation $e^{\theta i} = \cos(\theta) + \sin(\theta)i$ from a different mathematical starting point. And, from this equation, Euler derived *the* most famous, *the* most wondrous, *the* most mysterious equation in all of mathematics: $e^{i\pi} + 1 = 0$. But that story, and that derivation, will have to wait for another essay.[29]

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[28] Feynman, I:22-10.

[29] See www.biblicalchristianworldview.net/Mathematical-Circles/eulerCrownJewel.pdf