The Discovery of the Golden Section and Other Amazing Connections

by James D. Nickel

The Greek mathematicians of the Classical Age (ca. 600-300 BC) developed a host of fascinating connections/observations using both construction[1] and deduction. The goal of this essay is to explore one of those, the renowned Golden Section.[2] As we travel the road of their thought, we shall behold these vistas: the sides and diagonals of a regular pentagon, the Pythagorean Theorem, the diagonal of a square, $\sqrt{2}$ (i.e., irrational numbers), and an amazing connection to the Fibonacci sequence.

The Pythagoreans first discovered irrational numbers, also known as incommensurable segments, not in the study of a right triangle (i.e., the length of the diagonal of a unit square), *but through the study of the regular pentagon*. Early records indicate that the bearer of the bad news of incommensurates was Hippasus of Metapontum (ca. 5th century BC), a student of Pythagoras.[3] By virtue of his impious discovery, the Pythagoreans either threw him overboard in the Mediterranean at worst or banished him from their cult at best.

Let's consider the regular pentagon. By definition, a polygon is a closed figure in a two-dimensional plane having three or more sides. Also, each vertex of the polygon shares the endpoint of two adjacent sides. A pentagon is a polygon with five sides. The sides and interior angles of a regular pentagon are all equal (i.e., a regular polygon is both equilateral and equiangular).

Hippasus considered first how to construct such a pentagon and then he investigated its sides and diagonals.[4] The following construction, using a straightedge and compass, is not technically according to Greek "standards," but it does show that a regular pentagon exists.

First, we construct line segment AB (or $\overline{AB}$). $\overline{AB}$ will be one of the sides of the regular pentagon (A and B

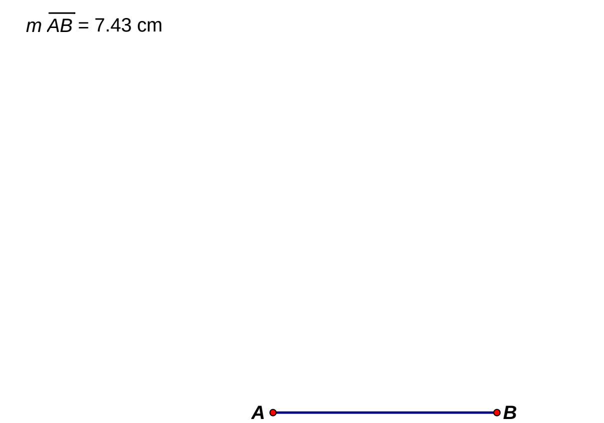

Figure 1

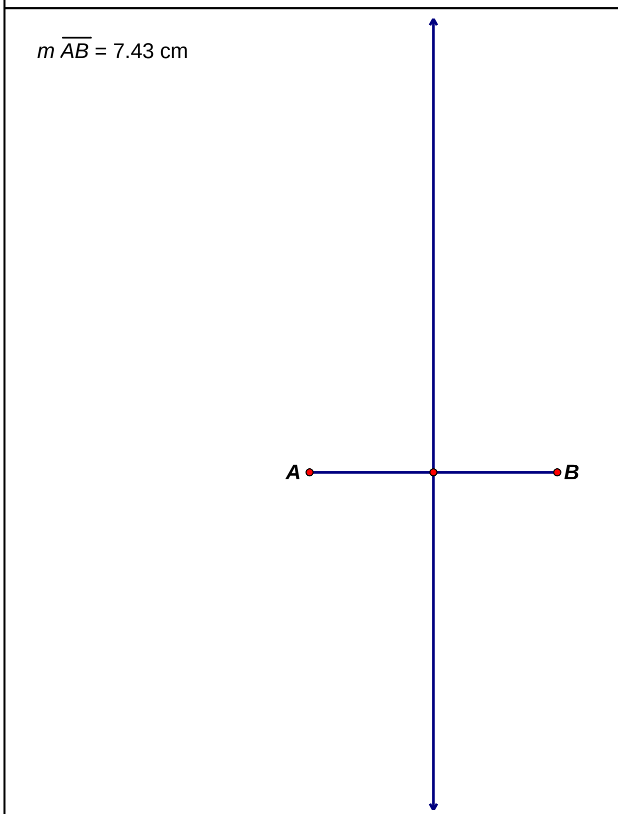

Figure 2

[1] The analysis of geometric figures using a straightedge and compass.
[2] I am following the summary arguments of Herbert Meschkowski, *Ways of Thought of Great Mathematicians: An Approach to the History of Mathematics* (San Francisco: Holden-Day, 1964), pp. 6-12. See also H. E. Huntley, *The Divine Proportion: A Study in Mathematical Beauty* (New York: Dover Publications, 1970).
[3] See Chalcis (ca. 283-330 AD), *On the Philosophy of the Pythagoreans* (Book I). The discovery of irrational numbers demolished the Pythagorean worldview that all is number (and, to them, number consisted of either positive integers or the ratio of positive integers, i.e., rational numbers).
[4] A diagonal of a polygon is a line segment that connects any two of its nonadjacent vertices.

are two of its five vertices). We let s = AB (the length of that side). In Figure 1, we have measured this length (something the ancient Greeks would never do).

Second, we construct the perpendicular bisector of $\overline{AB}$ (Figure 2).

Third, we construct a circle with center at B and radius = s (Figure 3).

Fourth, we consider a line segment of which $\overline{AB}$ is a part. We situate the end of this segment at A and let it intersect the perpendicular bisector and the circle at F and D respectively so that DF = s (Figure 4). Generally, Greek constructions do not allow for a "marked ruler." Their constructions use a straightedge with no marks. The Pythagoreans would have known of this distinction, but it is necessary to make it in this case to prove that a regular pentagon can be constructed. Note: AB = DF = s and D will be the third vertex of the regular pentagon.

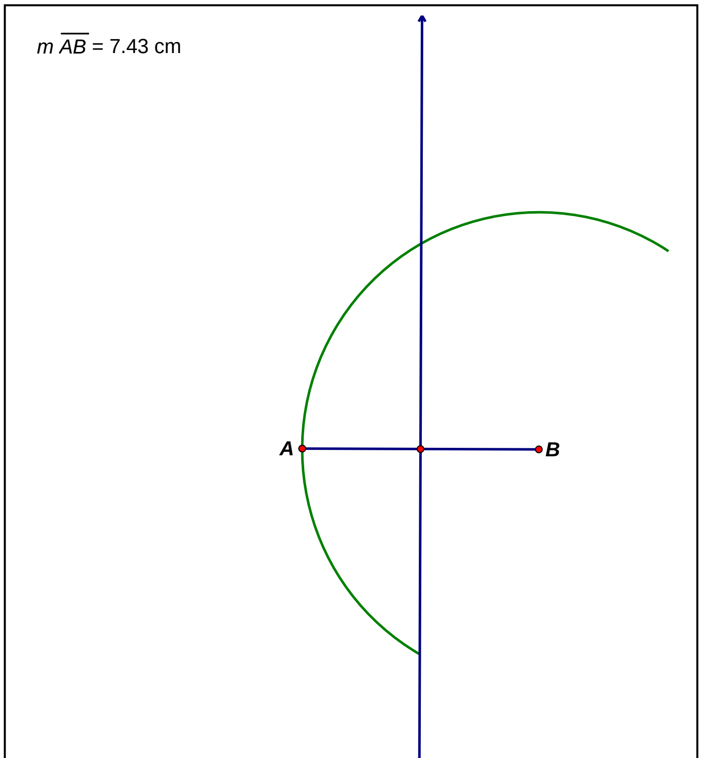

Figure 3

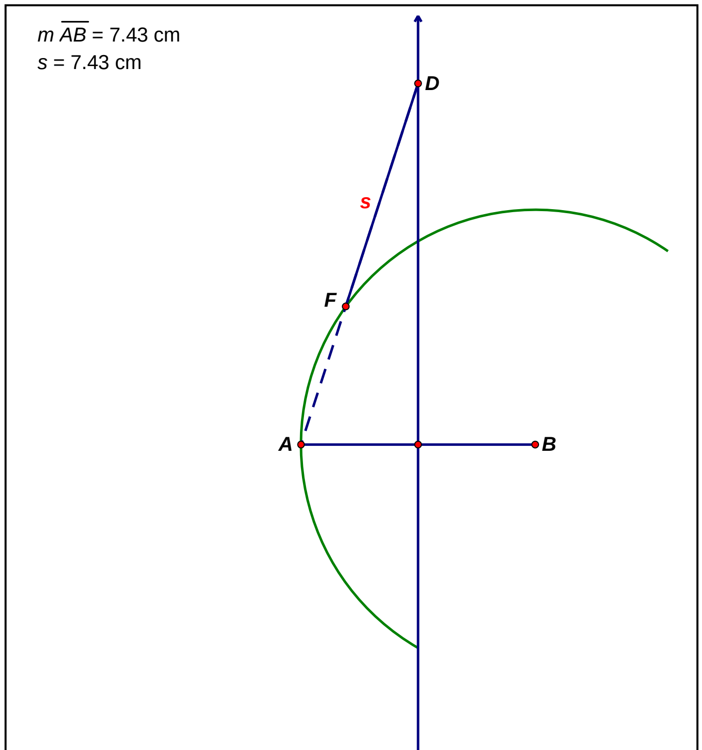

Figure 4

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Fifth, in similar fashion, we situate the end of this marked segment at A and let it intersect the perpendicular bisector and the circle at G and C respectively so that GC = s (Figure 5). Note: AB = DF = GC = s and G will be the fourth vertex of the regular pentagon.

Sixth, we construct a circle with center at A and radius = s (Figure 6).

Seventh, we construct a circle with center at D and radius = s (Figure 7). Next, we set E as the intersection of these two circles. This point is now the fifth vertex of the regular pentagon.

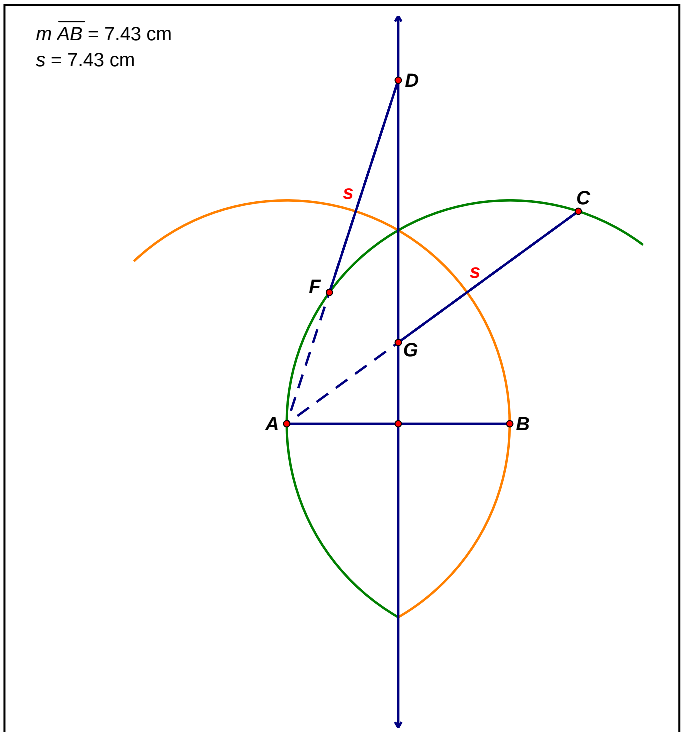


Figure 6

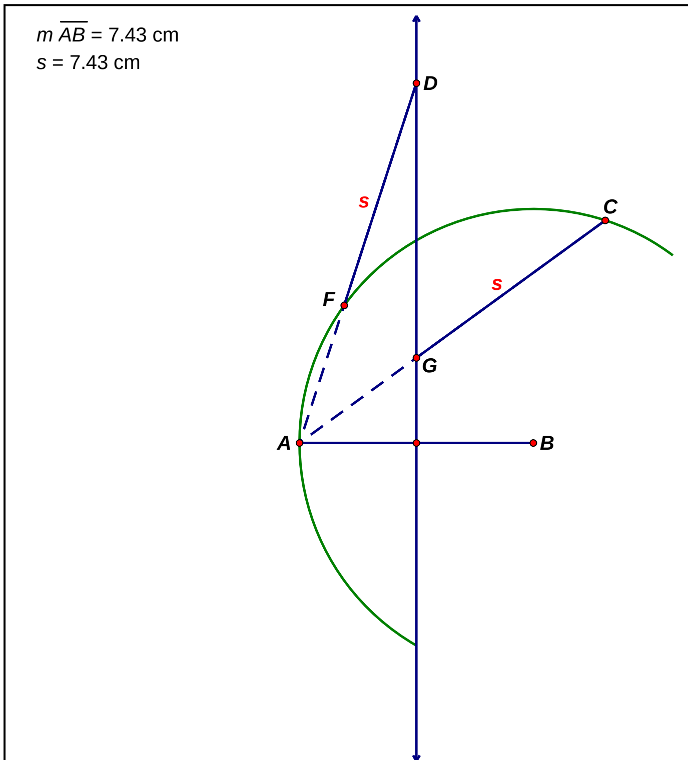


Figure 5

By construction, pentagon ABCDE is now regular (Figure 8).

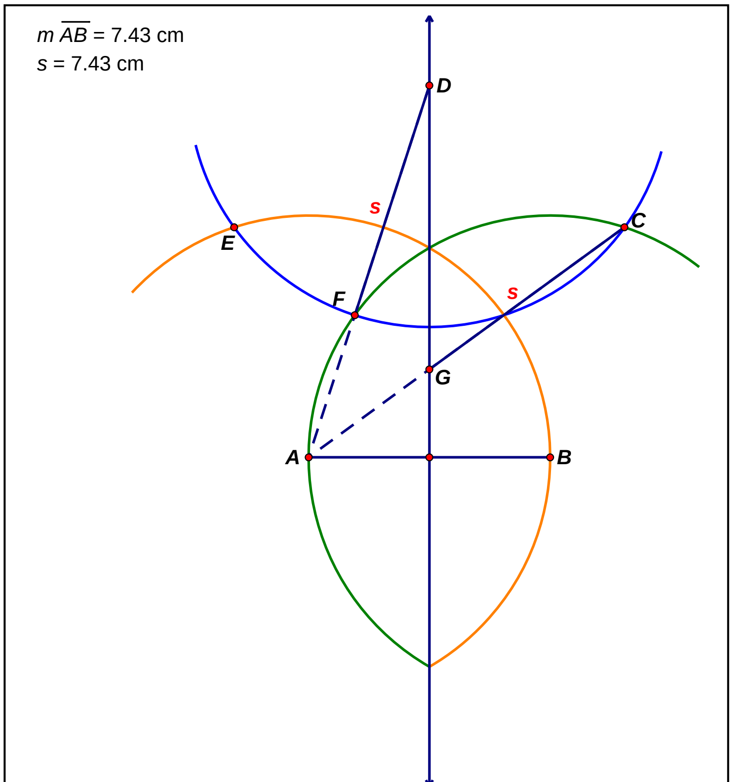


Figure 7

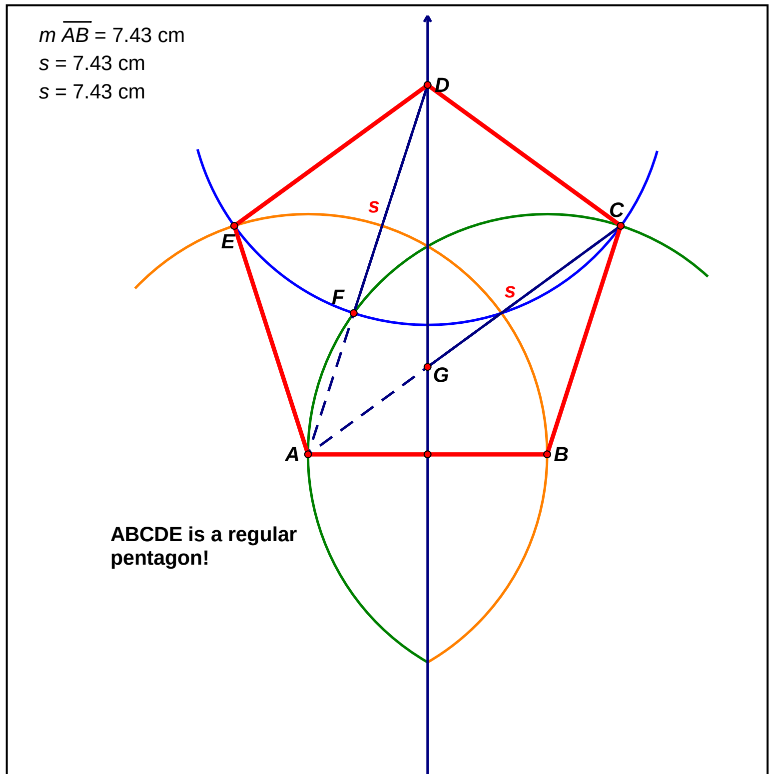


Figure 8

In Figure 9, we note a few diagonals of the regular pentagon, namely $\overline{EB}$, $\overline{AD}$, and $\overline{AC}$ where EB = AD = AC.

What is the relationship between these diagonals and the sides of the constructed pentagon? In the figure, a ratio is calculated to the hundredths place. This ratio (≈ 1.62) is the Golden Ratio (or Golden Section). Let d = diagonal; the Golden Section is φ (the Greek letter "phi") = $\frac{d}{s}$. Is this ratio rational? *Let's assume that it is and see if we can reason to a contradiction.* If we can, we will have done followed in the footsteps of Hippasus. And, by doing so, we will have constructed two segments that are incommensurate; i.e., *their ratio is irrational.* Let's stay clear of ships just to be safe while doing this!

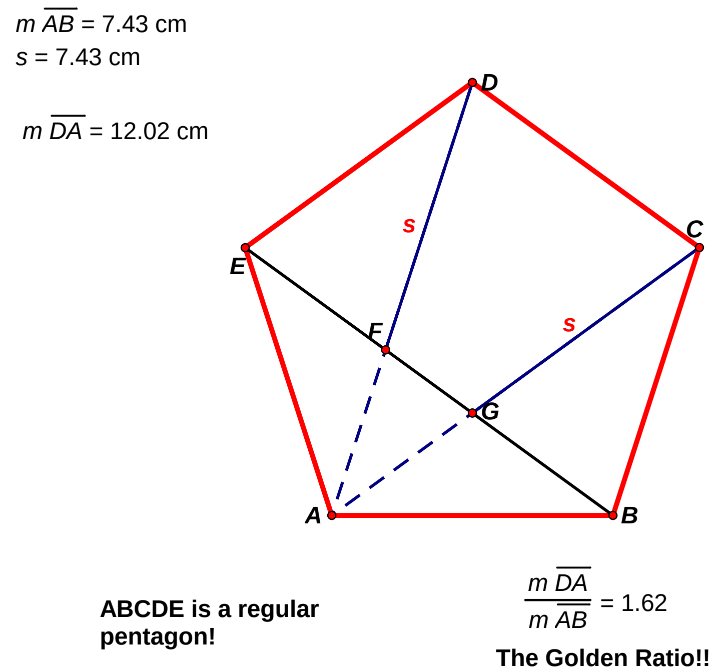


Figure 9

Let's construct a series of similar regular pentagons. We start with our construction. Let P_1 (first regular pentagon) be ABCDE. Next, we consider the diagonals $\overline{BD}$ and $\overline{AC}$ (Figure 10). We let d_1 (first diagonal) = AC and s_1 (first side) = AF. Hence, FC = $d_1 - s_1$. We now consider $\overline{FC}$ and $\overline{FB}$ (CF = FB = $d_1 - s_1$) and let them be the sides of a second regular pentagon P_2 such that the ratio of the corresponding sides of P_1 and P_2 are φ; i.e., $\frac{DE}{CF} = \frac{AE}{BF} = \varphi$ (Figure 11).

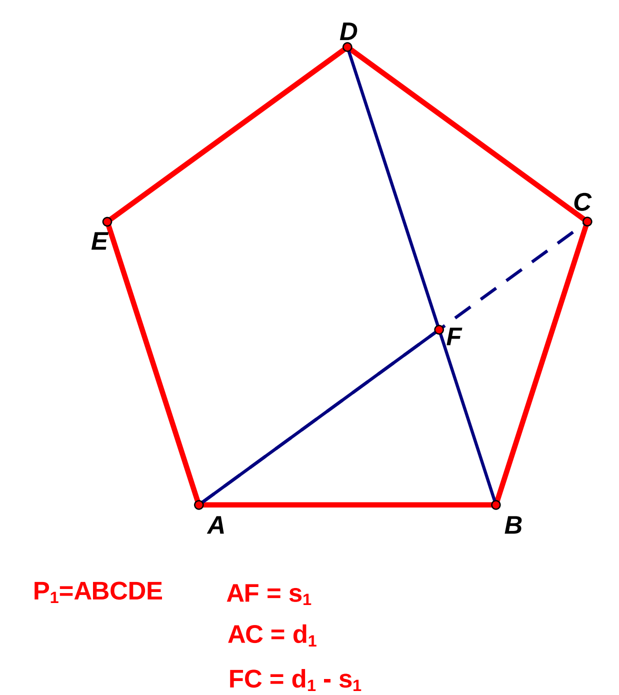
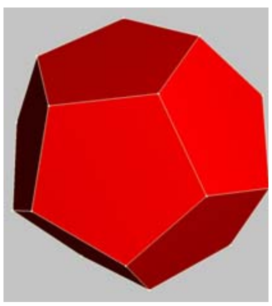


Figure 10

BGHCF is regular pentagon P_2. In P_2, $s_2 = d_1 - s_1$, $BH = d_2 = s_1$, and $JH = d_2 - s_2$. As before, we now consider $\overline{JH}$ and $\overline{JG}$ ($JH = JG = d_2 - s_2$) and let them be the sides of a third regular pentagon P_3 such that the ratio of the corresponding sides of P_2 and P_3 are φ; i.e., $\frac{CF}{JH} = \frac{FB}{JG} = \varphi$ (Figure 12).

GKLHJ is regular pentagon P_3. In P_3, $s_3 = d_2 - s_2$, $GL = d_3 = s_2$, and $ML = d_3 - s_3$. We can continue this process *ad infinitum*. Figure 13 shows the next step.



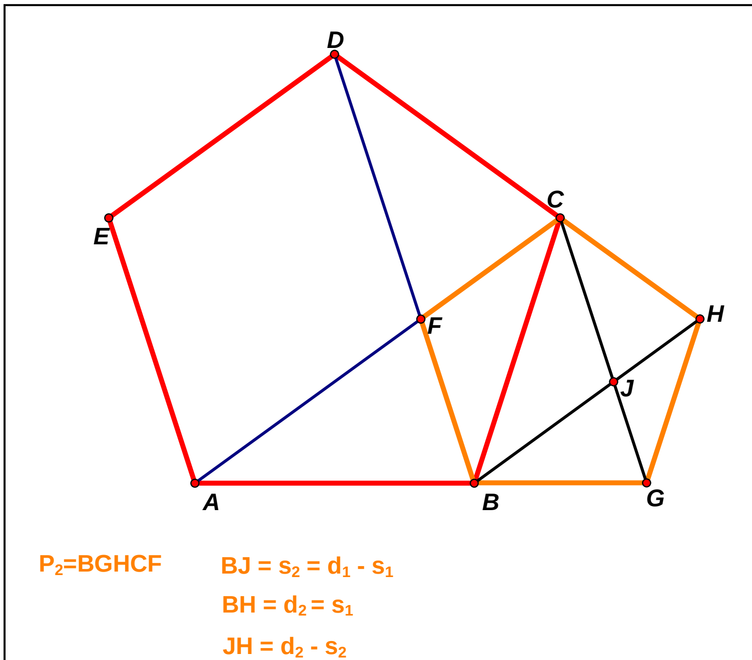


Figure 11

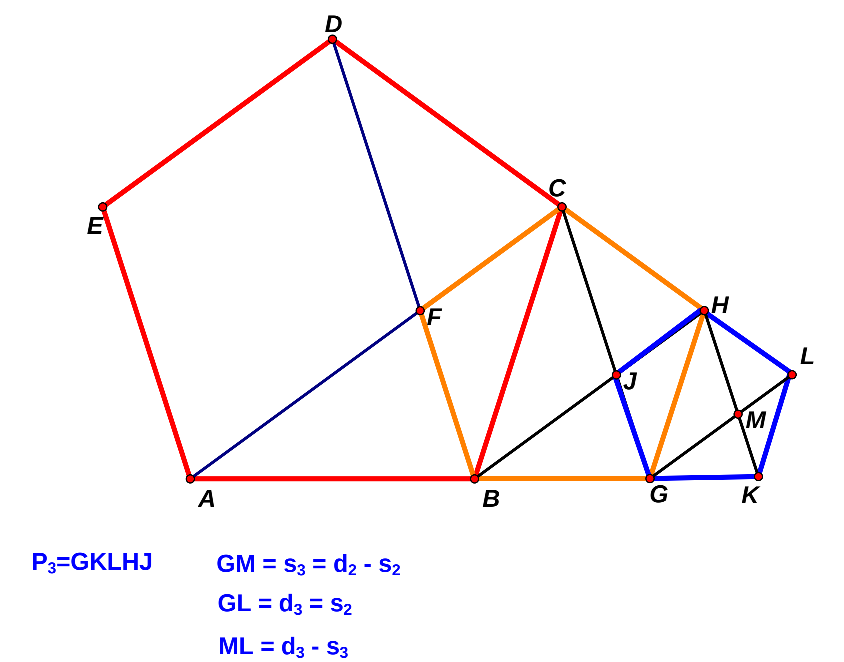


Figure 12

In general (from the pattern revealed), $s_n = d_{n-1} - s_{n-1}$ and $d_n = s_{n-1}$. Let's consider $\frac{d_1}{s_1} = \varphi$. If this ratio is rational, then $\frac{d_1}{s_1} = \frac{x_1 f}{y_1 f}$ when x_1 and y_1 are positive integers and *f* is a common factor. For example, $\frac{9}{33}$ is rational because $\frac{9}{33} = \frac{3 \cdot 3}{11 \cdot 3}$. We note this equivalency:

$$\frac{d_1}{s_1} = \frac{x_1 f}{y_1 f} \Leftrightarrow$$
$$d_1 = x_1 f \text{ and } s_1 = y_1 f$$

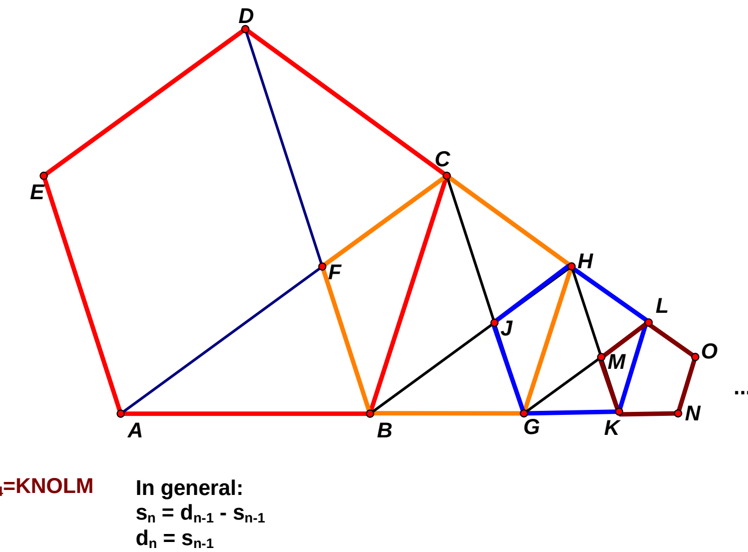


Figure 13

From the general pattern, we note $s_2 = d_1 - s_1$ and $d_2 = s_1$. By substitution, we get:

$$s_2 = x_1 f - y_1 f = f(x_1 - y_1)$$
$$d_2 = y_1 f = x_2 f \text{ (for some integer } x_2\text{)}$$

By similarity observations, we note that $y_2 < y_1$ and $x_2 < x_1$. Repeating this argument, we can conclude, in general:

$$y_{n+1} < y_n \text{ and } x_{n+1} < x_n$$

This process must end after a *finite* number of steps, since for any *k*, x_k and y_k are positive integers. This conclusion contradicts the fact that we can continue to construct smaller similar regular pentagons *ad infinitum*. Hence, our assumption is false and $\frac{d_1}{s_1} = \varphi$ must be irrational; i.e., d_1 and s_1 are incommensurate.

Anyone game for a cruise in the Mediterranean?

By the same reasoning (except we start with a square and construct smaller similar squares), you can prove that the ratio of the diagonal *d* of a square to its side *s* is incommensurate. In Figure 14, this ratio is *approximately* 1.41 (d = AC). Using the Pythagorean Theorem, it is exactly:

$$d = \sqrt{4^2 + 4^2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

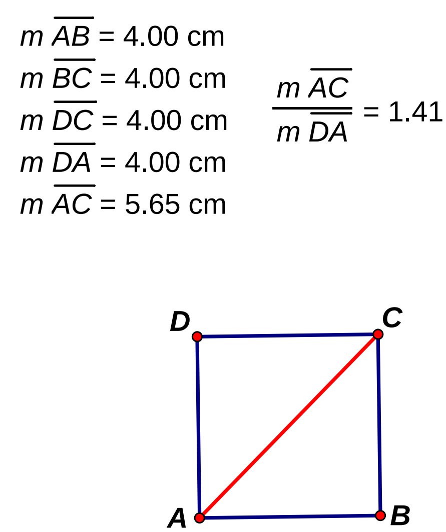


Figure 14

Without taking the time to do the construction, we can consider square S_n with side s_n and diagonal d_n. We define S_{n+1} as the square whose side is $d_n - s_n$. For the next square (similar and smaller), we note these relationships:

$$s_{n+1} = d_n - s_n$$
$$d_{n+1} = s_n - s_{n+1}$$

From these two equations, we can prove, duplicating the regular polygon logic, that side s_n and diagonal d_n are incommensurate.

What is to keep us from going from a smaller square to a larger similar square? Nothing. Let's see what happens. We just interchange n with n+1. We get (Note: we are changing the order above so we can substitute):

$$d_n = s_{n+1} - s_n \Leftrightarrow$$
$$s_{n+1} = d_n + s_n$$
$$s_n = d_{n+1} - s_{n+1} \Leftrightarrow$$
$$d_{n+1} = s_n + s_{n+1}$$

By substitution:

$$d_{n+1} = s_n + d_n + s_n = 2s_n + d_n$$

Now, we can technically construct larger and larger squares. Given a starting square S_1 of side *s* and diagonal *d* as "unit," we can construct the larger and similar S_2, S_3, S_4, *ad infinitum*. The reason for starting with a "unit" *s* and *d* is entirely Pythagorean. The Pythagoreans embraced a world view that saw "unit" starting points as a seed that generated other mathematical objects that carried that "unit" as a primary characteristic. We can easily see this with the counting numbers (positive integers): {1, 2, 3, …}. The seed is 1 (the unit) and each successive number is 1 more than the number that precedes it.

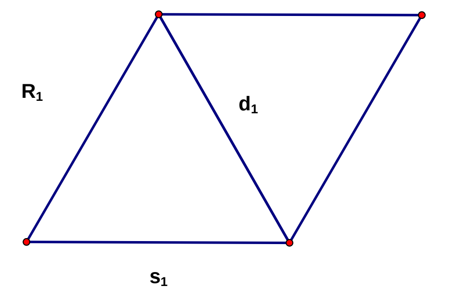


Figure 15

Let's see what happens if d = s = 1. Constructing this situation generates a special quadrilateral called a rhombus (Figure 15). Let's use our results above to construct the next rhombus in the chain:

$$s_2 = 1 + 1 = 2$$
$$d_2 = 2(1) + 1 = 3$$

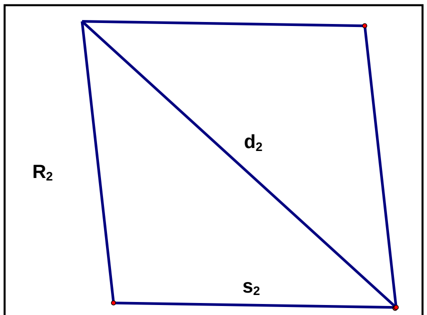


Figure 16

We get Figure 16. The third rhombus is looking more and more like a square (Figure 17).

$$s_3 = 2 + 3 = 5$$

$$d_3 = 2(2) + 3 = 7$$

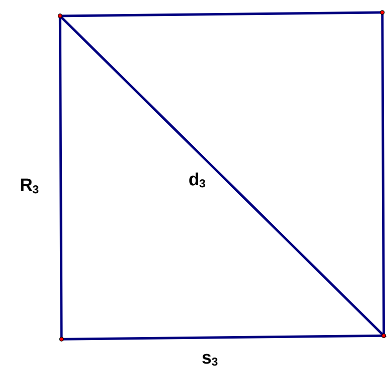


Figure 17

We can continue with the process and we will get a square at "infinity." This table shows us what is happening:

	d_n	s_n	$\frac{d_n}{s_n}$
R_1	1	1	1
R_2	3	2	1.5
R_3	7	5	1.4
R_4	17	12	$1.41\overline{6}$
R_5	41	29	1.4137931

This table shows us that as *n* gets larger (or approaches infinity; i.e., in symbols, $n \to \infty$), $\frac{d_n}{s_n}$ is converging to a specific number (it appears to be *irrational)*. But, as $n \to \infty$, $\frac{d_n^2}{s_n^2}$ approaches a *rational* number.

There is more to this table; i.e., there is another pattern hidden in it. Since we are talking about squares, consider d_n^2 and s_n^2. Observe:

	d_n	s_n	d_n^2	s_n^2
R_1	1	1	1	1
R_2	3	2	9	4
R_3	7	5	49	25
R_4	17	12	289	144
R_5	41	29	1681	841

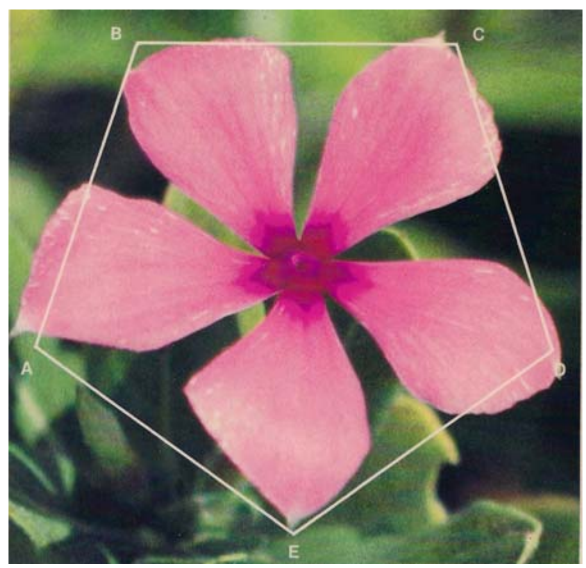
Look at the fourth column and compare it to the third column. If we double s_n^2, and subtract this value from the associated value in the third column ($2s_n^2 - d_n^2$), we get an alternating sequence of 1 and -1:

	d_n	s_n	d_n^2	s_n^2	$2s_n^2$	$2s_n^2 - d_n^2$
R_1	1	1	1	1	2	1
R_2	3	2	9	4	8	-1
R_3	7	5	49	25	50	1
R_4	17	12	289	144	288	-1
R_5	41	29	1681	841	1682	1

Since $(-1)^n = -1$ when *n* is odd and $(-1)^n = 1$ when *n* is even, we can generate this relationship[5]:

$$2s_n^2 - d_n^2 = (-1)^{n+1}$$

Also, as $n \to \infty$, $\frac{d_n}{s_n} \to \sqrt{2}$ and $\frac{d_n^2}{s_n^2} \to 2$. Or, using limit notation, $\lim_{n\to\infty} \frac{d_n}{s_n} = \sqrt{2}$ (meaning the ratio of the diagonal of a square to its side is incommensurate) and $\lim_{n\to\infty} \frac{d_n^2}{s_n^2} = 2$. Again, the Pythagorean Theorem confirms this. If the side of a square is 1, then its diagonal is $\sqrt{2}$ or $1^2 + 1^2 = 2$.



According to the German mathematician S. Heller, the Pythagoreans may have discovered the Golden Section using this argument applied to the regular pentagon.[6] Review the proof that the ratio of the diagonal to the side of a regular pentagon is incommensurate. Note that we employed a sequence of smaller and smaller regular pentagons. As we did with the square, let's reverse the process; i.e., let's start with a "unit" pentagon and make subsequent ones larger and larger. Given $s_n = d_{n-1} - s_{n-1}$ and $d_n = s_{n-1}$, we again interchange n with n – 1 and get:

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[5] We can invoke a method of proof called mathematical induction to establish this relationship with certainty. I am bypassing this method of proof and establishing the relationship by simply noting the pattern. Mathematical induction is commonly attributed to the French theologian/mathematician Blaise Pascal (1623-1662), although it appears that the Pythagoreans were aware of the concepts governing this method.

[6] S. Heller, *Die Entdeckung der stetigen Teilung durch die Pythagoreer*, Abhhandlungen der D. Ak. D. Wiss. , Klasse f. Math., Phys., u. Technik, No. 6, v. 54 (1958).

$$s_{n-1} = d_n - s_n \Leftrightarrow$$
$$d_n = s_{n-1} + s_n$$
$$s_n = d_{n-1} \Rightarrow \text{(by substitution)}$$
$$d_n = s_{n-1} + d_{n-1}$$

We can now use this to construct an infinite sequence of pentagons with increasing sides and diagonals. As with the square, we set our unit pentagon using s = d = 1. We then get a sequence of pentagons (P_1, P_2, P_3, …) that appear, as $n \to \infty$, to approach a regular pentagon (Figure 18).

Entering our results in a table, we discover an amazing sequence of numbers.

	s_n	d_n
P_1	1	1
P_2	1	2
P_3	2	3
P_4	3	5
P_5	5	8
P_6	8	13
P_7	13	21
P_8	21	34
P_9	34	55
P_{10}	55	89

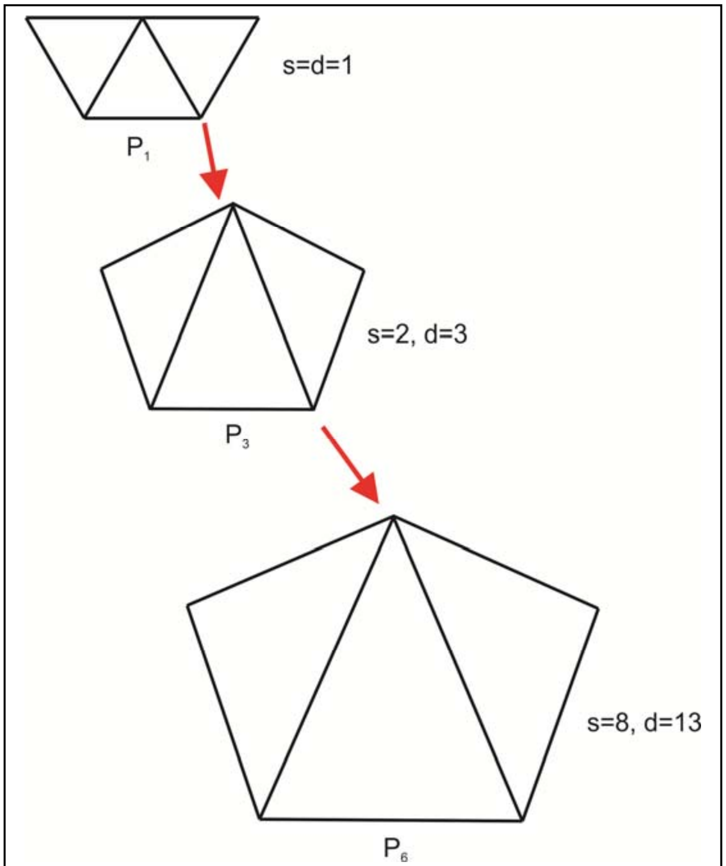


Figure 18

First, let's inspect this table and see if we can generate an alternating sequence of 1, -1, 1, -1, as we did with the squares. First, let's add the column s_n^2

	s_n	d_n	s_n^2
P_1	1	1	1
P_2	1	2	1

	s_n	d_n	s_n^2
P_3	2	3	4
P_4	3	5	9
P_5	5	8	25
P_6	8	13	64
P_7	13	21	169
P_8	21	34	441
P_9	34	55	1156
P_{10}	55	89	3025

To generate an alternating sequence of 1, -1, 1, -1 working with s_n and d_n requires a little bit of ingenuity, but not much. If we subtract s_n from d_n and multiply the difference by d_n, we get it:

	s_n	d_n	s_n^2	$d_n(d_n - s_n)$	$d_n(d_n - s_n) - s_n^2$
P_1	1	1	1	0	-1
P_2	1	2	1	2	1
P_3	2	3	4	3	-1
P_4	3	5	9	10	1
P_5	5	8	25	24	-1
P_6	8	13	64	65	1
P_7	13	21	169	168	-1
P_8	21	34	441	442	1
P_9	34	55	1156	1155	-1
P_{10}	55	89	3025	3026	1

Before we proceed to see what $d_n(d_n - s_n) - s_n^2 = (-1)^{n+1}$ means, the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … in this table show us a surprising connection of the generation of the regular pentagon to the terms of the famous Fibonacci Sequence.[7] The connection of these numbers to the Golden Section will be revealed shortly.

Now, back to $d_n(d_n - s_n) - s_n^2 = (-1)^{n+1}$ and its meaning. Remember, this expression is about the diagonals and sides of a pentagon. We consider the n^{th} sequence of the constructed pentagons, i.e., P_n.

[7] The sequence is named after the Italian mathematician Leonardo Fibonacci (ca. 1170-ca. 1250), who first noticed their appearance in his study of the population of rabbits and the genealogy of bees.

Consider a side and diagonal such that $s_n < d_n$. Divide diagonal d_n into two segments, s_n and $d_n - s_n$. We can now construct a rectangle of sides d_n and $d_n - s_n$. Geometrically, $d_n(d_n - s_n) - s_n^2 = (-1)^{n+1}$ means the difference between the area of this rectangle is alternatively 1 more or 1 less than the area of the square of sides s_n.

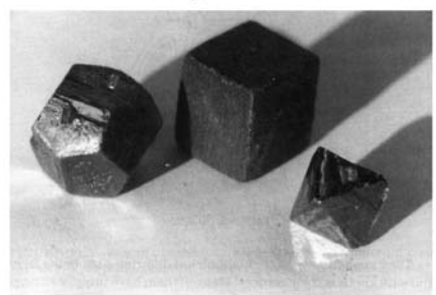
In review, when we generated a square after an finite number changes to the unit rhombus (s = d = 1), we derived the relationship $2s_n^2 - d_n^2 = (-1)^{n+1}$ and finally, at infinity, $\frac{d^2}{s^2} = 2$ or $d^2 = 2s^2$. We have a similar situation here. When we generated a regular pentagon after a finite number changes to the unit pentagon (s = d = 1), we derived the relationship $d_n(d_n - s_n) - s_n^2 = (-1)^{n+1}$, and finally, at infinity, $d(d - s) = s^2$.

As any student of geometry would immediately see, $d(d - s) = s^2$ is the definition of the Golden Section.[8] It comes from this proportion:

$$\frac{d}{s} = \frac{s}{d - s}$$

Geometrically, this proportion states that the ratio of *d* to *s* (the whole to its longer part) is equal to the ratio of *s* to *d* – *s* (its longer part to its shorter part). This is also called the "division of a line into extreme and mean ratio." In terms of our rectangle and square, it means that if a diagonal *d* is divided into two segments, *d* and *d* – *s*, such that the area of this rectangle is equal to the area of the square of sides *s*. In this case, $\frac{d}{s} = \frac{s}{d - s} = \varphi$, *the Golden Section!*

Pyrite



Using algebra, we can determine the exact value of φ. Since $\frac{d}{s} = \varphi \Leftrightarrow d = \varphi s$, then, by substitution, we get:

$$\frac{\varphi s}{s} = \frac{s}{\varphi s - s}$$

Factoring, we get:

$$\frac{\varphi s}{s} = \frac{s}{s(\varphi - 1)}$$

Cancelling, we get:

[8] Euclid effectively proved this relationship in Book IV, Proposition 10, where he showed that each base angle of an isosceles triangle is double the third or vertex angle (i.e., we have a 72°-72°-36° triangle). We see this triangle (ΔACD) in Figure 9 where the Golden Section is revealed: $\frac{AC}{DC} = \frac{AD}{DC} = \frac{d}{s} = \varphi$.

$$\varphi = \frac{1}{\varphi - 1}$$

Multiplying both sides by $\varphi - 1$, we get:

$$\varphi(\varphi - 1) = 1$$

Expanding the left side, we get:

$$\varphi^2 - \varphi = 1$$

Gathering all the terms on the left side, we get:

$$\varphi^2 - \varphi - 1 = 0$$

Applying the Quadratic Formula, we get:

$$\varphi = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Indeed, the Quadratic Formula shows that $\frac{d}{s}$ is incommensurate. Since we only concerned with the positive solution (we are dealing with distances), we get the exact/approximate value of the Golden Section:

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

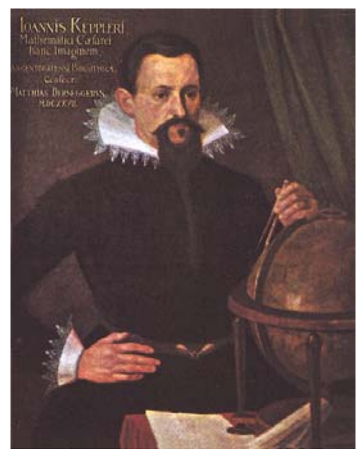
We now return to the Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …. Let F_n be the n^{th} term of the sequence. What happens to the ratio $\frac{F_{n+1}}{F_n}$ and *n* gets larger and larger? Inspect the table:

n	$\frac{F_{n+1}}{F_n}$	n	$\frac{F_{n+1}}{F_n}$
1	$\frac{1}{1} = 1$	7	$\frac{21}{13} \approx 1.615$
2	$\frac{2}{1} = 2$	8	$\frac{34}{21} \approx 1.619$
3	$\frac{3}{2} = 1.5$	9	$\frac{55}{34} \approx 1.618$

n	$\frac{F_{n+1}}{F_n}$	n	$\frac{F_{n+1}}{F_n}$
4	$\frac{5}{3} \approx 1.67$	10	$\frac{89}{55} \approx 1.618$
5	$\frac{8}{5} = 1.6$	11	$\frac{144}{89} \approx 1.618$
6	$\frac{13}{8} = 1.625$	12	$\frac{233}{144} \approx 1.618$

Amazingly, $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} \approx 1.618 = \varphi$!

By these derivations, it is no wonder why the German mathematician and astronomer Johannes Kepler (1571-1630) reflected, "Geometry has two great treasures: one is the theorem of Pythagoras; the other the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."[9]



In conclusion, let's review the vistas we have beheld. First, we established a way to construct a regular pentagon. Then, we used a nesting algorithm (to make smaller and smaller similar regular pentagons) to prove that the ratio of the diagonal of a regular pentagon to its side is incommensurate (or irrational). Then, we turned our attention to the construction of a square by reversing the nesting algorithm and applying it an infinite number of times to a "unit" rhombus (d = s = 1). At infinity, we concluded that $d^2 = 2s^2$ or $\frac{d}{s} = \sqrt{2}$ for any square of side *s* and diagonal *d*. Using this reverse algorithm as a guide, we constructed a regular pentagon by applying it an infinite number of times to a "unit" pentagon (d = s = 1). We then discovered an interesting table of numbers that connects the construction of a regular pentagon to terms of the Fibonacci sequence. At infinity, we concluded that $d(d - s) = s^2$ for any regular pentagon of side *s* and diagonal *d*. Then, we noted that $d(d - s) = s^2$ is derived from the proportion $\frac{d}{s} = \frac{s}{d-s}$. Setting this proportion equal to φ $\left(\frac{d}{s} = \frac{s}{d-s} = \varphi\right)$, we calculated $\varphi = \frac{1+\sqrt{5}}{2}$, the Golden Section!

[9] Cited in Huntley, p. 23.

The Discovery of the Golden Section and Other Amazing Connections

by James D. Nickel

Our logic followed the paths of the Classical Greeks, particularly the Pythagoreans, and thereby provides background to many statements *merely given* in elementary Geometry and Algebra. Not many students know that the source of "the division of a line into its extreme/mean ratio" is in the construction of the regular pentagon! I have also added graphics throughout these demonstrations (e.g., a soccer ball, a dodecahedron, a sand dollar, pyrite crystals, and a flower) to demonstrate that these abstract ideas can be connected to creational objects, one man-made (but well before the time of the Pythagoreans) and the others God-made. The multiple applications of the Fibonacci sequence to creational patterns intensify these connections.

For the Biblical Christian, this unity in diversity (abstract principles, or general patterns, revealed in the concrete, or particular, objects) exists because the Triune God, the ultimate unity in diversity, is the author of both. Unlike the Pythagorean worldview ("all is number"), the Biblical Christian worldview is not founded upon any aspect of the creation; it is founded upon the God of creation. Hence, the relationships explored in this essay lead the Biblical Christian, not to the worship of creation, but the worship of its Author.