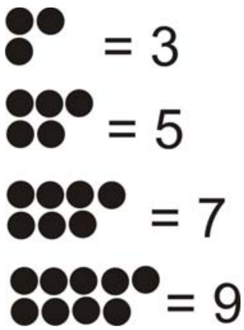
The Conic Sections: Amazing Unity in Diversity

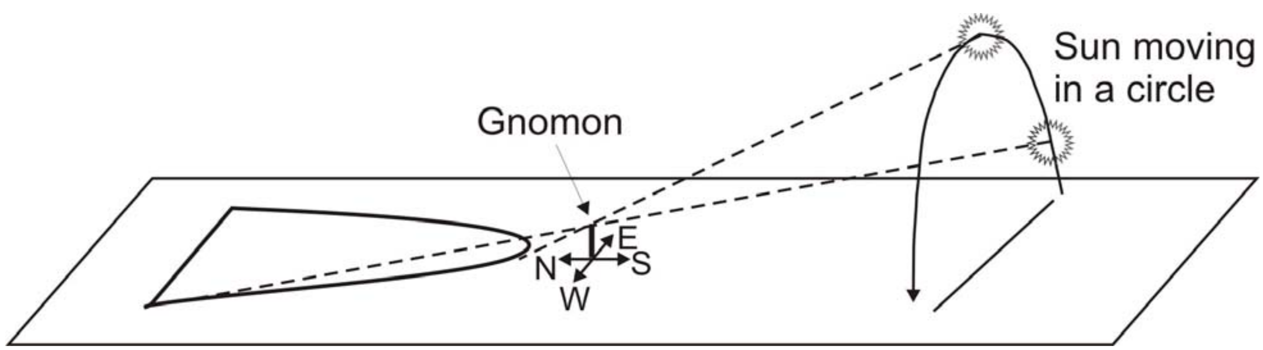
One of the most incredible revelations about quadratic equations in two unknowns is that they can be graphed as a circle, a parabola, an ellipse, or a hyperbola. These graphs, albeit quadratic functions, are very different from each other. Some are finite and closed (the circle and the ellipse; the circle is a special form of the ellipse where the major axis equals the minor axis). Some wander off to infinity (the parabola and the hyperbola). Three are in one piece (the circle, the parabola, and the ellipse). One is in two pieces (the hyperbola).

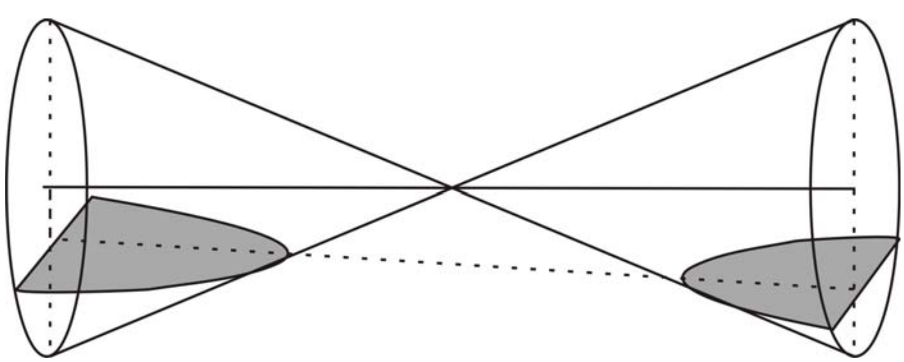
Is there a family likeness in these curves? Is there a unity in this diversity? Yes, the ancient Greeks were the first to analyze these curves and they found this unity. According to historian Otto Neugebauer (1899-1990), "In antiquity the conic sections are needed for the theory of sundials and I have conjectured that the study of these curves originated from this very problem."[1] The sundial, in its simplest form, is an upright stick called a *gnomon* (meaning "knows") whose shadow moves with the Sun. The gnomon knows (or measures) solar time. This stick has become identified with any shape that looks like the capital Greek letter Γ (gamma). The gnomon is in the form of a carpenter's square. The Greek mathematician Pythagoras (ca. 582-ca. 500 BC) pictured odd numbers in the form of dots that resembled a gnomon.



In the Balkan Peninsula, the Sun is always in the southern sky (never directly overhead). Therefore, you can model the Sun's movement by a circle in a vertical plane offset to the south (see the figure). As the Sun rises and sets, a branch of a hyperbola is traced out on the horizontal plane of the Earth by the tip of the shadow of the gnomon.

This model suggests the shape of an ice-cream cone on its side. The top of the gnomon is the *apex* (or vertex) of the cone. To make what is called a "two-cusped" cone, what the Greek mathematician Apollonius (ca. 262-ca. 190 BC) used, you mirror image (or reflect) the first cone about its apex. The model now depicts both branches of a hyperbola.





[1]Otto Neugebauer, *The Exact Sciences in Antiquity* (New York: Dover Publications, [1957] 1969), p. 218, 226.

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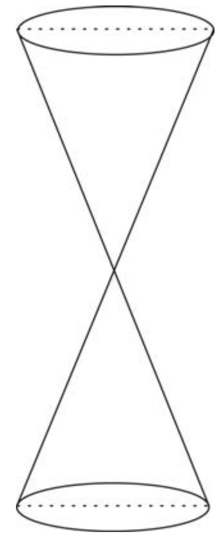
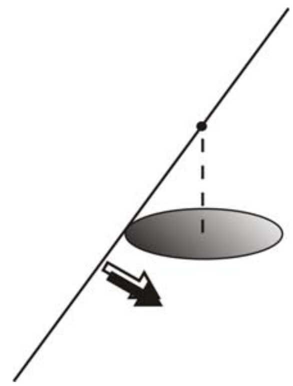
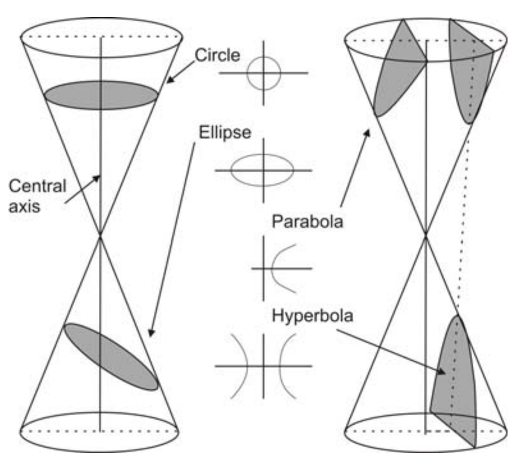


The “family name” for the four curves (circle, parabola, ellipse, and hyperbola), *conic sections*, comes from this cone shape. Think of the traditional ice cream cone and you have a good image of this three-dimensional figure. As we have noted, the sharp point or tip at the bottom of the ice cream cone is the apex of the cone. Three of the conic sections (circle, ellipse, and parabola) can be generated from a *single* cone. The hyperbola is generated from the two-cusped cone. We can think of *generating* the two-cusped cone by first imaging a horizontal disk and a slanting line segment (called the *generatrix*) leaning over the center of the disk and touching the disk at one point (i.e., it is tangent to the disk). Now imagine taking this line segment and rotating it around the disk while holding it fast at the point (apex) lying above the center of the disk. This rotation will generate a cone above and below this apex (the generated cone, in this instance, is called the *directrix*). The two individual cones are called *nappes*[2] (singular nappe).



If we intersect these two cones with a plane in various positions, the four conic sections appear along the edges of the truncated pieces.

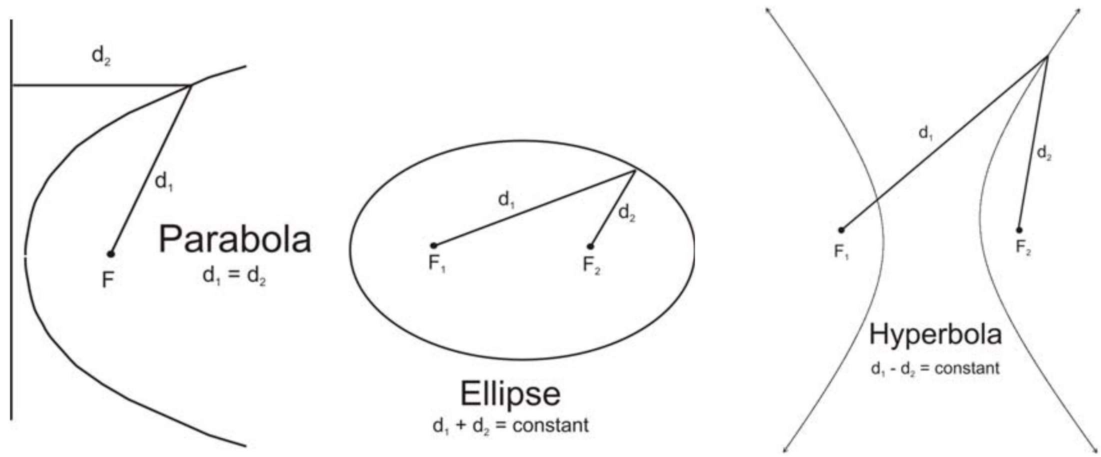
A plane cutting the cone at an angle	*Conic section*
orthogonal (at a right angle) to the central axis of the cone	circle
between right angles to the central axis of the cone and parallel to the generatrix	ellipse
parallel to the generatrix	parabola
between a plane parallel to the generatrix and a plane parallel to the central axis of the cone	hyperbola



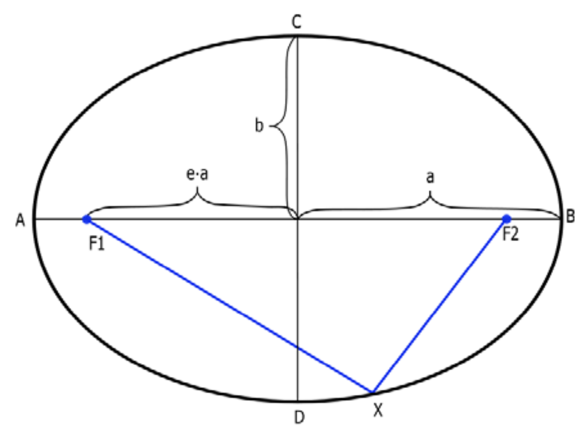
[2] *Nappe*, in French, means “sheet or surface.”

If we think of the locus of points for each conic section, we have:

The curve of	*is the locus of*
the circle	points that have the *same distance* the center of the circle.
the ellipse	points whose distances to two given foci have a constant sum.
the parabola	points that have the *same distance* to a focus and the directrix
the hyperbola	points whose distances to two given foci have a *constant difference*.



An *ellipse* is a regular oval and it can be traced by a point moving in a plane in such a way that the sum of its distances from two other fixed points is constant. Referring to figure, when you add the distances F_1X and F_2X you will always obtain the same answer for as the long as the figure is an ellipse.



Woodworkers use this property of an ellipse. With two nails, non-stretchable string and a pencil they can draw an ellipse (used when making tabletops, etc.). F_1 and F_2 represent the position of the nails. The string is tied around these nails, looped around a pencil (X) and then the ellipse can be drawn. Because the string does not stretch, it can maintain the ellipse's property of a constant sum of lengths between two fixed points.

A *circle* is a perfectly round plane figure whose circumference is at all times equidistant from its circumference. You can easily construct a circle with a compass. Viewing a rainbow from an airplane will reveal the fact that it is a circle. From Earth we see it as a semicircle.

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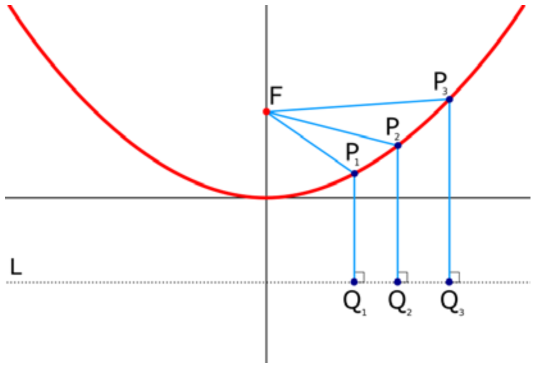
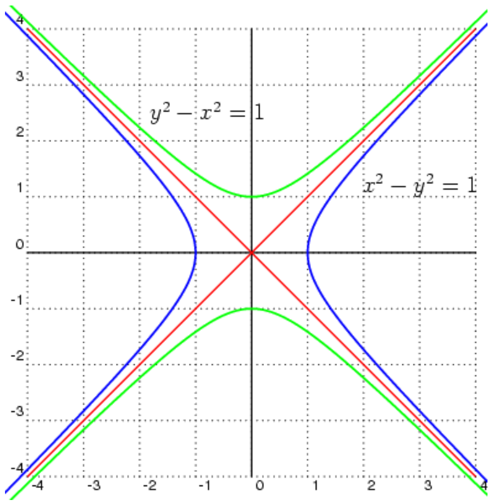
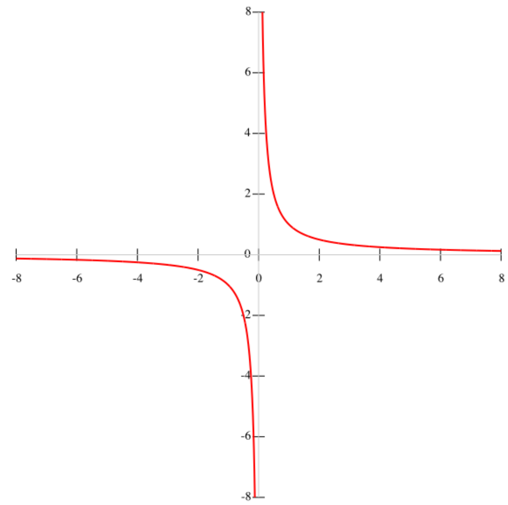


The *parabola* is the locus of points P such that the distance from a line (called the directrix) to P is equal to the distance from P to a fixed point F (called the focus); i.e. $FP_1 = FQ_1$, etc. The line L is called the directrix and F is the focus point.

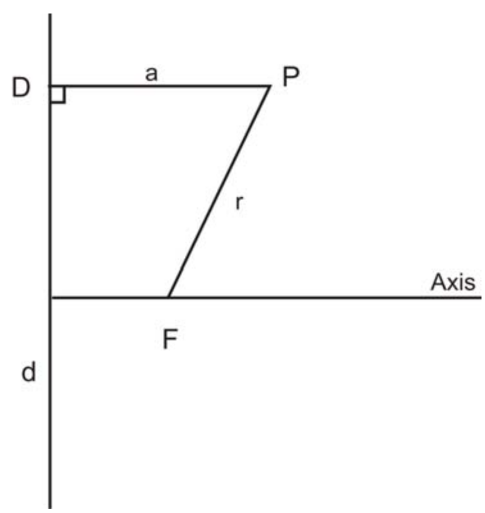
The *hyperbola* is a plane curve with two equal, infinite branches.





xy=1

Another way to define the conic sections is in terms of *eccentricity*.[3] Given the focus, F, the directrix *d*, and a point on the curve P, eccentricity, *e* is defined by the following ratio where r = PF and a = PD.



$$e = \frac{r}{a}$$

For a parabola, since $r = a$, then $e = 1$. For an ellipse, since $r < a$, then $e < 1$. For the hyperbola, since $r > a$, then $e > 1$. It is this eccentricity that provides the etymological foundation for the respective names of these three conic sections:

e	*Curve*	*Meaning*
e = 1	parabola	equal to
0 < e < 1	ellipse	to fall short
e > 1	hyperbola	in excess of

Since a circle is not "off center," its eccentricity is 0 ($e = 0$). Since r = the radius of the circle, then, for e = 0, a (the distance from P to the directrix) must be *at infinity*. In symbols, using the limit concept of the Calculus:

[3] Eccentricity comes from the Greek word meaning "out of or off center."

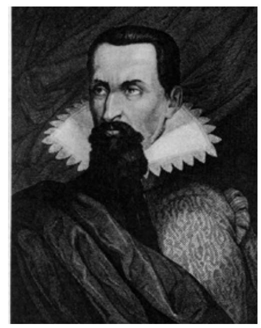
$$\lim_{a \to \infty} \frac{r}{a} = 0$$

Since an ellipse with eccentricity of 0 becomes a circle, the full qualification for the possible values of *e* for an ellipse is $0 < e < 1$.

These curves have another geometric property called reflection, a property that is used to concentrate or reflect light rays or sound beams. The Greek mathematician Archimedes (ca. 287-212) used a parabolic mirror to concentrate the rays of the Sun.

Curve	*Reflects rays coming from the focus*
circle	back to the center of the circle
ellipse	into the other focus
parabola	as a parallel outgoing ray or beam
hyperbola	as if coming from the other focus

Because of the reflective property of parabolas, they are used in automobile headlights or searchlights (the mirror in each headlight has a curved surface formed by rotating a parabola about its axis of symmetry). If we place a light at the focus of the mirror, it is reflected in rays parallel to the axis. In this way, a straight beam of light is formed. The opposite principle is used in the giant mirrors of reflecting telescopes, radar transmitters, solar furnaces, sound reflectors, and in satellite or radio wave dishes. With these instruments the beam comes toward the parabolic surface and is brought into focus at the focus point.

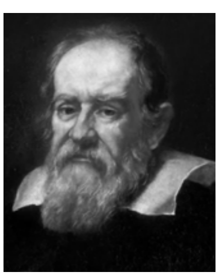


Johannes Kepler (Public Domain)

The reflective property of an ellipse is used in the creation of what are called "whispering galleries." If you are at one focal point in a room shaped in the size of an ellipse, you will be able to hear the whispers of a person located at the other focal point (even if other conversations are going on in the room).

In the 17th century, the German astronomer and mathematician Johannes Kepler (1571-1630), augmented the heliocentric cosmology of the Polish astronomer Nicholas Copernicus (1473-1543) by showing that the planets orbit the Sun in a curve that resembles an ellipse (where the Sun is at one focus). When a projectile is launched from the Earth with a velocity between the orbital velocity of the Earth (about 17,000 miles per hour) and escape velocity (about 25,000 miles per hour), its path will resemble an ellipse.

The path of an electron follows an elliptical path as it whirls around the nucleus of an atom. In this ordered dervish, the appearance of pentagons is evident.



Galileo Galilei (Public Domain)

The Italian mathematician Galileo Galilei (1564-1642) showed that the motion of a projectile follows the path of a parabola (facing down). When a projectile is launched from the Earth with a velocity less than the orbital velocity of the Earth, its path will be parabolic.

The hyperbola is also known as a "shock wave" curve. When an airplane flies faster than the speed of sound, it creates a shock wave heard on the ground as a "sonic boom." The shock wave has a shape of a cone with its apex located at the front of the airplane. This wave intersects the ground in the shape of the hyperbola. When a projectile is launched from the Earth with a velocity greater than escape velocity, its path will resemble the hyperbola. The hyperbola is also essential to understanding LORAN (stands for LOng RAnge Navigation), a technique used for locating ships at sea. The Cassegrain telescope makes special use of both parabolic and hyperbolic mirrors.

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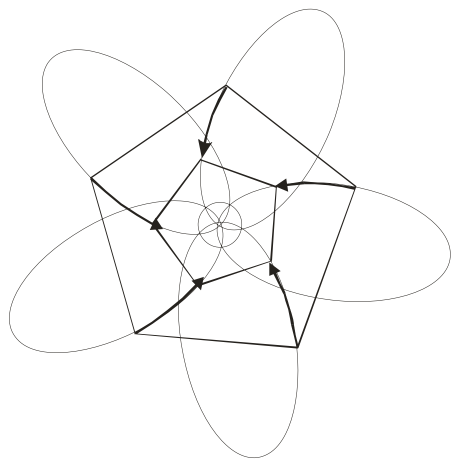
The circle has so many applications in our world that to list them all would take a book.[4]

The ancient Greeks understood the conic sections as useful in some ways, but playful toys in many other ways. They could never have envisioned, even in a dream, how useful and essential their playful games would become. It is crucial to note that the quadratic equations reflecting the geometric behavior of the conic sections serve two purposes in God's world:

1. They are essential for an understanding of the harmonious order of God's creative design and His sustaining word of power (Psalm 104:24; Colossians 1:15-17, Hebrews 1:3).
2. They serve as tools that enable us to take effective dominion over the creation (Genesis 1:26-28).

I hope that your study of conic sections helps you to both appreciate and enjoy the mathematical wonders of God's world as revealed by the conic sections.



[4] See Ernest Zebrowski, *A History of the Circle: Mathematical Reasoning and the Physical Universe* (London: Free Association Books, 1999).