

# COUNT THE NUMBER OF LEGS AND DIVIDE BY 4

BY JAMES D. NICKEL

The 19<sup>th</sup> century mathematician Carl Friedrich Gauss (1777-1855) was known both for his mathematical trait of *curiosity* and his renowned ability to calculate. His ability to calculate can be illustrated by a boyhood experience. One day in class, Gauss's math teacher must have needed a break from the stresses and strains of the day so he decided to give his class some busy work (of course, we know that no teacher ever gives "busy work" to his or her students). He wrote this problem on the blackboard, "Add up all the numbers from 1 to 100." As soon as he removed his chalk from the blackboard, the teacher sat down at his desk thinking confidently, "Ah, some rest, finally. This ought to keep these pesky children busy for some time." No luck, Mr. Teacher, you forgot that Carl Friedrich Gauss was in your class. Within seconds, student Gauss got up, walked forward, and placed his slate face down on the teacher's desk. Incredulously, the teacher turned the slate over and saw written on it just one number: 5,050. Gauss was right, but how did he do it so fast?<sup>1</sup>

Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.

Here is Gauss's reasoning, "I considered the sum as follows:  $1 + 2 + 3 + 4 + 5 + \dots + 96 + 97 + 98 + 99 + 100$ . Then I noticed that  $1 + 100$  (the first and the last number in the sum) = 101. Then I noted that  $2 + 99$  (the second and second to last number in the sum) = 101. Continuing,  $3 + 98 = 101$ ,  $4 + 97 = 101$ ,  $5 + 96 = 101$ , etc. The last sum is  $50 + 51 = 101$ . After 50 such additions, the sums taken from the beginning and the end finally meet in the middle. Hence, I have 50 sums all totaling 101.  $50 \times 101 = 50 \times (100 + 1) = (50 \times 100) + (50 \times 1) = 5000 + 50 = 5050$ . QED."<sup>2</sup>

Here is another version of the type of problem that Gauss solved. Consider the sum of the first 9 numbers:  $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$ . We see the same sort of pattern as Gauss encountered summing up the first 100 numbers with one important difference. Since we are summing an odd number of terms<sup>3</sup>, we do not get a "constant" number of sums that equal 10. The process works until we get to the middle number in the series, 5. It sort of stands there by itself. Since it stands by itself, can we discover a "sum rule" around it? We have seen this:  $1 + 9 = 10$ ,  $2 + 8 = 10$ ,  $3 + 7 = 10$ ,  $4 + 6 = 10$ . We've got four tens and 5 more. Therefore, our sum is 45. How does 45 relate to 5? The obvious fact that stands out is that  $\frac{45}{5} = 9$ .

Do you see a connection? There are nine terms and the "middle" term is 5.  $9 \times 5 = 45$  (the sum of the nine terms). To show how this will always work (for the sum of any odd number of consecutive terms), rewrite the series as follows:

$$\begin{aligned} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 &= 45 \\ 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 &= 45 \end{aligned}$$

Notice that with the first and last terms we added 4 to the first and then we subtracted 4 from the last (both terms sum to 10). We then added 3 to the second term, then we subtracted 3 from the eighth term (both terms sum to 10). We added 2 to the third term and then we subtracted 2 from the seventh term (both terms sum to 10). We added 1 to the fourth term and then we subtracted 1 from the sixth term (both terms

<sup>1</sup> According to Eric Temple Bell in his book *Men of Mathematics* (New York: Simon and Schuster, [1937, 1965] 1986), the instructor gave Gauss this problem to solve:  $81,297 + 84,495 + 81,693 + \dots + 100,899$  where the common difference between consecutive terms is 198 and the number of terms is 100. Summing the numbers from 1 to 100 uses the same principle and is simpler.

<sup>2</sup> QED is the abbreviation for the Latin phrase *quod erat demonstrandum* meaning "which was to be demonstrated." Mathematicians like to use this abbreviation to show that, having reached the end of a proof, what they wanted to demonstrate is proven.

<sup>3</sup> Each number in the addition problem  $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$  is technically called a *term*. Each of the members of this expression, a series of quantities, or the like, is called a term.

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sum to 10). By this method we have rewritten the given series as a multiplication problem (nine groups of 5).

Here we see two principles illustrated: one for the sum of an even number of consecutive terms and the other for the sum of an odd number of consecutive terms. Is there a way that we can unite these two procedures? We can, if we use a somewhat novel approach. Consider this joke. Mr. City Slicker visits a farm pasture, notices a herd of cows and asks Mr. Farmer, "How many cows do you have?" Mr. Farmer replies, "137." Mr. City Slicker asks, "How did you figure that out?" Mr. Farmer responds, "Why, Mr. City Slicker, it's easy. I just counted the number of legs and divided by 4!"

We will use the Mr. Farmer "find the number of cows" principle to figure out the sum of either an even number or odd number of consecutive terms. Here's how we do it. Let's consider summing  $1 + 2 + 3 + 4 + 5 + 6$  and  $1 + 2 + 3 + 4 + 5 + 6 + 7$ . Let's write down each addition problem twice:

$$\begin{array}{r} 1 + 2 + 3 + 4 + 5 + 6 \\ 6 + 5 + 4 + 3 + 2 + 1 \\ \hline 1 + 2 + 3 + 4 + 5 + 6 + 7 \\ 7 + 6 + 5 + 4 + 3 + 2 + 1 \end{array}$$

Now, sum up the numbers by columns. The first group:  $7 + 7 + 7 + 7 + 7 + 7$  or 6 groups of 7; i.e.,  $6 \times 7 = 42$ . The second group:  $8 + 8 + 8 + 8 + 8 + 8 + 8$  or 7 groups of 8; i.e.,  $7 \times 8 = 56$ . Since the numbers 42 and 56 are twice the sum of each respective series, all we need to do is to divide by 2 to get the sum of each series. Hence, the sum of the first series is  $42/2 = 21$  and the sum of the second series is  $56/2 = 28$ .

This is the rule that we can discern in both cases: *sum the first and the last term, multiply this sum by the number of terms, and then divide that answer by 2.*

Let's try out this rule on the series Gauss solved. Sum the first and last terms:  $1 + 100 = 101$ . Multiply by the number of terms:  $101 \times 100$ . Then divide the answer by 2. Instead of multiplying 101 by 100 first, we will divide 100 by 2. Since this equals 50, all we need to do is to calculate  $101 \times 50$  (which is what Gauss did) to get 5050.

Let's not stop here. This rule works for summing consecutive numbers. Will it also work for sums of numbers that succeed each other by equal steps? For example, consider  $4 + 8 + 12 + 16 + 20$ . Note that every succeeding term is 4 more than the previous term. Take the sum of the first and last terms:  $4 + 20 = 24$ . Multiply 24 by the number of terms (5 in this case).  $24 \times 5 = 100$ . Divide 100 by 2 to get 50. Check it out. Does it work? You bet it does. Try this rule out on several other examples:

1.  $3 + 9 + 15 + 21 + 28 + 34 =$  (ans. 111)
2.  $7 + 14 + 21 + 28 + 35 + 42 =$  (ans. 147)
3.  $20 + 25 + 30 + 35 + 40 =$  (ans. 150)
4.  $15 + 18 + 21 + 24 + 27 + 30 =$  (ans. 135)

What we have been considering are examples of an *arithmetic series*. An arithmetic series is defined as the *sum* of a sequence of numbers where each number is separated by a constant interval or constant difference. In general, an arithmetic sequence (list) is defined as follows:

$$a, a + d, a + 2d, \dots, a + (n - 1)d$$

The letter  $d$  represents the "common difference" between each term. The simplest arithmetic sequence is the set of natural numbers: 1, 2, 3, ...,  $n$ . In this case,  $d = 1$ . Algebraically, How would we determine  $1 + 2 + 3 + \dots + n$ ? First, we let  $S$  be equal to this sum. We get:

$$S = 1 + 2 + 3 + \dots + n$$

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As we did with our above examples, we can also write the same sum *in reverse*. We get:

$$S = n + (n - 1) + (n - 2) + \dots + 1$$

Adding these two equations, we get:

$$2S = n(n + 1)$$

Why? Note:

$$S = 1 + 2 + 3 + \dots + n$$

$$S = n + (n - 1) + (n - 2) + \dots + 1$$

By adding, we get  $2S$  on the left. Since each column on the right add up to  $(n + 1)$  and there are  $n$  columns, we get  $n(n + 1)$ .

Dividing both members of the equation  $2S = n(n + 1)$  by 2, we get:

$$S = \frac{n(n + 1)}{2}$$

This formula, named the *Gaussian sum* in honor of Gauss, only works for summing the first  $n$  terms of the set of natural numbers. Notice that this formula fits our rhetorical analysis above: *sum the first and the last term* ( $n + 1$ ), *multiply this sum by the number of terms* ( $n$ ), *and then divide that answer by 2*. How can we prove this rhetorical rule, in general, for *any* arithmetic sequence (not just the set of natural numbers) with any common difference (not just  $d = 1$ )? As before, we let  $S$  be equal to this sum. Using our general definition of an arithmetic sequence, we let  $d =$  the common difference,  $a_1 =$  the value of the first term, and  $a_n =$  the value of the last term. We get:

$$S = a_1 + (a_1 + d) + (a_1 + 2d) + \dots + (a_n - 2d) + (a_n - d) + a_n$$

We now write the same sum *in reverse*. We get:

$$S = a_n + (a_n - d) + (a_n - 2d) + \dots + (a_1 + 2d) + (a_1 + d) + a_1$$

Adding these two equations, we get:

$$2S = n(a_1 + a_n)$$

This result should be easy to see. Dividing both members of this equation by 2, we get:

$$S = \frac{n(a_1 + a_n)}{2}$$

We have proved our rhetoric; i.e., *sum the first and the last term, multiply this sum by the number of terms, and then divide that answer by 2*. Hence, to find the sum of any arithmetic sequence, all we need to know is the first term, the last term, and the total number of terms. We do *not* need to know the common difference.

Let's see how this works given the following series:

$$5 + 8 + 11 + 14 + 17 = ?$$

Given the above formula,  $S = \frac{5(5 + 17)}{2}$ , the parenthesis bracketing 5 and 17 means we add these two numbers first. Let's do it:  $5 + 17 = 22$ . Then we multiply  $22 \times 5 = (20 + 2)5 = 100 + 10 = 110$ . Now, we

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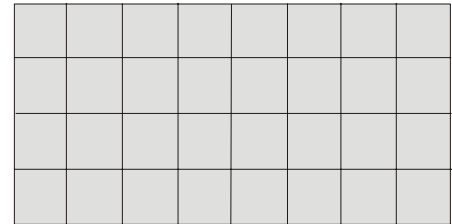
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divide  $\frac{110}{2} = 55$ . We could also apply the distributive rule to generate the same answer:  $5(5 + 17) = 5 \times 5 +$

$17 \times 5 = 25 + (10 + 7)5 = 25 + 50 + 35 = 110$ . Therefore,  $\frac{110}{2} = 55$ . Either approach (do the calculation within the parenthesis first or apply the distributive rule) generates the same answer.

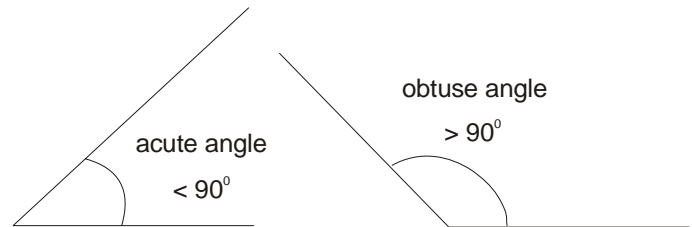
What is interesting about this “find the number of cows” principle is that it appears in other branches of the mathematics. This is not unique. An in depth study of mathematics will unveil many unifying principles that connect a diversity of topics. Finding “unity in diversity” is one of the joys of doing mathematics and this discovery reflects upon the wisdom of the triune God revealed in the Lord Jesus Christ. It is in Christ, the wisdom and logos of God (read Colossians 2:3; John 1:1-3), that the discipline of mathematics consists (or coheres; read Colossians 1:15-17 again). We would therefore expect to find faint glimpses of this wisdom of God in mathematics. One example of this wisdom is finding unifying principles in a diversity of mathematical propositions.

Given a *rectangle*, we can find its area by multiplying its length by its width. Let’s say that the dimensions of the rectangle below are 4 inches by 8 inches. Finding the area is the same as counting how many one-inch squares there are in this bounded area.  $8 \times 4 = 32$ . The area is 32 square inches.

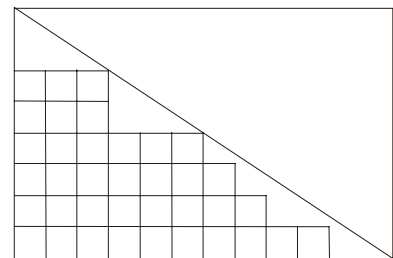


Note also that the adjacent sides of a rectangle form a *right angle* (90 degrees) with each other. This is another way of saying that the adjacent sides of a rectangle are *perpendicular*<sup>4</sup> to each other. Most homes are built on the principle of perpendicularity because it provides excellent stability for the structure.

An angle less than (symbol:  $<$ )  $90^\circ$  ( $^\circ$  is the symbol for degrees) is called an *acute*<sup>5</sup> *angle* and an angle greater than (symbol  $>$ )  $90^\circ$  is called an *obtuse* *angle*.<sup>6</sup> Both angles would require extra support if a building contractor decided to use them as walls in a house. A figure bounded by three lines is called a *triangle*. A right triangle has one angle that measures  $90^\circ$  while the other two angles will always be acute (but the sum of both will always equal  $90^\circ$ ).



Let’s consider a right triangle. How would we calculate its area? Let’s say that the triangle pictured measures 8 inches in height with a base of 12 inches. If we try to count the number of one-inch squares within its bounded surface, we will run into trouble as illustrated.



Why? Because of the two acute angles. The problem is easily solved by applying the “find the number of cows” principle. If we cannot calculate the area of one triangle, then let’s calculate the area of two, get an answer, and then divide that answer by 2.

<sup>4</sup> Perpendicular comes from a Latin word meaning “plumb line.” A carpenter uses a plumb line to make sure his work is square or perfectly vertical ( $90^\circ$ ) to the horizon.

<sup>5</sup> Acute is Latin for “sharpen” which has a word meaning “needle” at its root.

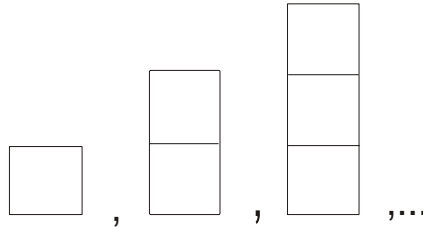
<sup>6</sup> Obtuse is Latin for “blunt.”

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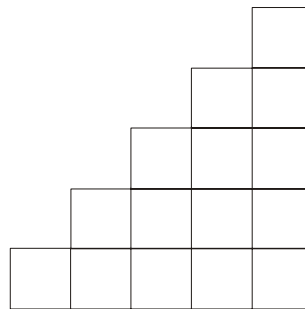
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In this case, the area of the two triangles is the area of one rectangle. A rectangle of length 8 inches by 12 inches has an area of 96 square inches. Hence, the area of the triangle with a height of 8 inches and a base of 12 inches is 48 square inches. In general, if  $A = \text{area}$ ,  $b = \text{base}$ , and  $h = \text{height}$ , then the area of a triangle is determined by the following formula:  $A = \frac{bh}{2}$ .

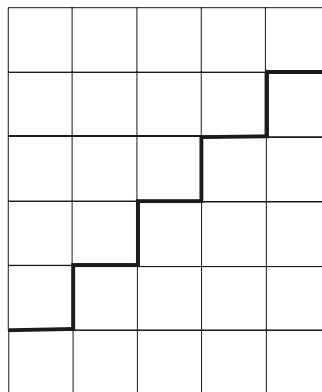
In conclusion, ancient Greek mathematicians liked to picture numbers. They did this because the base 10 number system restricted them when it came to calculations and in the development of algebra. For example, 1, 2, 3, ... could be pictured as follows:



So, the sum,  $1 + 2 + 3 + 4 + 5$ , could be pictured in the form of a “stair step” triangle.



Now, we rotate this figure  $180^\circ$  fitting another “stair step” triangle on top of the first one.



Using the decimal system, what we have done in pictures is:

$$1 + 2 + 3 + 4 + 5$$

$$5 + 4 + 3 + 2 + 1$$

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The sum (now pictured geometrically as area) is calculated as before:  $6 \times 5 = \frac{30}{2} = 15$ . We have now demonstrated the “find the number of cows” principle in three ways: arithmetically, algebraically, and geometrically. What a wonderful display of *unity in diversity*.