9.9 Musical Means

A primary measure of Statistics is the mean, the average, or, to be mathematically precise, the arithmetic mean, symbolized by the Greek letter μ. The arithmetic mean (Lesson 6.15) of two numbers a and b is:

Terms & Concepts Introduced
1. Diatonic scale
2. Frequency (Physics)
3. Fundamental tone (tonic)
4. Harmonic mean
5. Harmonic overtone
6. Harmonic sequence
7. Harmonic Triangle
8. Musical fifth
9. Musical fourth
10. Octave
11. Pythagorean scale
12. Weighted average

$$\mu = \frac{a+b}{2}$$

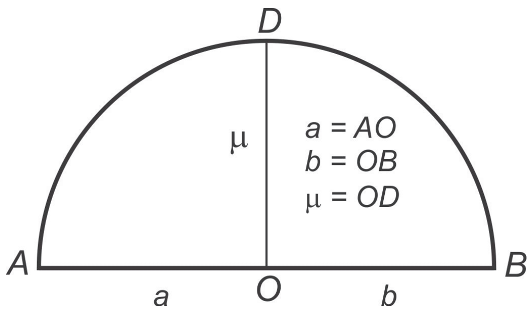

Figure 1. The arithmetic mean of a and b where $a = b$.

The arithmetic mean or three numbers a, b and c is:

$$\mu = \frac{a+b+c}{3}$$

In general, the arithmetic mean of n numbers a_1, a_2, ... , and a_n is:

$$\mu = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

With a straightedge and compass, we can construct the arithmetic mean of two positive numbers (Figure 1). In the figure, the radius r is the diameter of a circle divided by 2:

$$\mu = r = \frac{a+b}{2}$$

In Figure 1, $a = b = r$. If $a \neq b$ (Figure 2), we connect the unequal segments into one line segment and then bisect that line segment. (See Lesson 5.2 homework.)

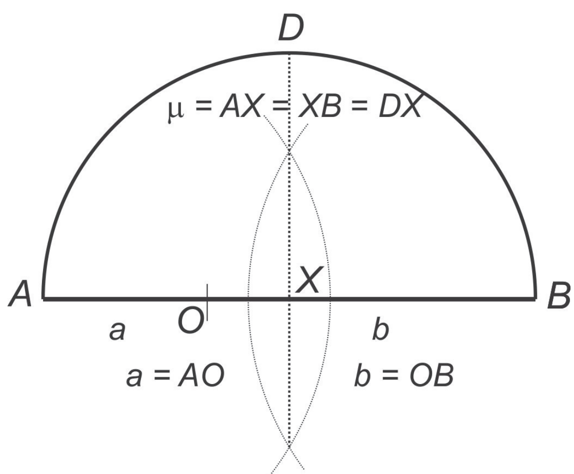

Figure 2. The arithmetic mean of a and b where $a \neq b$.

Figure 3. Herman von Helmholz. Source: Public Domain.

The Musical Harmony of Pythagoras

Since Pythagoras (6th century BC) joined music[1] with mathematics, there is another mean related to music, the **harmonic mean**. Let's first briefly explore the development of musical scales to understand the nature of this mean.

[1] Pythagoras probably noticed the harmonies in the ringing tones of hammers striking an anvil. He then determined that the weight of the hammers was responsible for the relative notes. The credit of connecting lengths of strings with the frequency

In 1863, the German physicist Herman von Helmholz (1821-1894) discovered that every music note emitted by musical instruments consists of a complex sound composed of a **fundamental tone** (or tonic) accompanied by other tones in harmony with it, the **harmonic overtones**. The frequencies of these harmonics are integral multiples of the fundamental tone. When we pluck a given string, it produces a note with a frequency measured in vibrations per second. Let's say that a fundamental tone is 100 Hz (vibrations per second). Its overtones will be $2 \cdot 100$, $3 \cdot 100$, $4 \cdot 100$, etc. (Table 1).

Table 1	
Fundamental Tone (Hz)	**Overtones (Hz)**
100	
	200
	300
	400
	500
	600
	etc.

When we pluck a string at half its length, we produce its octave vibrating at twice the frequency. In music, when a string is halved, the note that we hear is an **octave** higher.[2] The sound an octave above 100 Hz is 200 Hz with overtones revealed in Table 2.

Table 2: Octave	
Fundamental Tone (Hz)	**Overtones (Hz)**
200	
	400
	600
	800
	1000
	1200
	etc.

The overtones in Table 2 are a subset of the overtones in Table 1. This means that when the 200 Hz is played, our ears hear part of the 100 Hz sound; i.e., there is an interpenetration telling our senses that there is a relationship between the 100 Hz sound and the sound an octave higher.

A similar perichoresis occurs, although less marked, for the **fifth**, so named because the higher note is just five notes above the lower on a given scale. Starting with the C note, the fifth is the G note (C-D-E-F-G). By experimentation, Pythagoras discovered that the interval between these two notes have frequencies in the ratio of 3 to 2 or:

$$\frac{\text{G(Hz)}}{\text{C(Hz)}} = \frac{3}{2}$$

If the frequency of C is 24 Hz and we let x = the frequency of G in Hz, we can solve for x:

(the number of occurrences within a given time) of their vibrations goes to Galileo Galilei (ca. 17th century), although some historians believe it belongs to Pythagoras. See Lloyd W. Taylor, *Physics: The Pioneer Science* (New York: Dover Publications, 1941), 1:376. In Statistics, frequency is the ratio of the number of actual to possible occurrences of an event. In Physics, **frequency** is the rate at which a vibration occurs.

[2] In Latin, *octava*, means "eighth day." In music, an octave is a sequence of eight notes occupying the interval between, and including, two notes, one having twice or half the frequency of vibration of the other.

$$\frac{x}{\cancelto{12}{24}\ \text{Hz}} = \frac{3}{\cancelto{1}{2}} \Leftrightarrow x = 36\ \text{Hz}$$

The overtones of each note are listed in Table 3.

Table 3	
C (Hz)	**G(Hz)**
24	36
48	**72**
72	108
96	**144**
120	180
144	**216**
168	252
192	**288**
216	324
240	**360**
264	396
288	432
312	etc.
336	
360	
etc.	

Do you see that alternate overtones of G (2^{nd}, 4^{th}, 6^{th}, etc.), highlighted in **bold**, match every third overtone of C (3^{rd}, 5^{th}, 7^{th}, 10^{th}, etc.), also highlighted in **bold**?

If we let C′ be the note one octave above C, we can commence our construction (Table 4) of the Pythagorean musical scale.

Table 4							
C				G			C′
24				36			48

Next, let's compute F such that C′ is a fifth above it. Therefore:

$$\frac{C'(\text{Hz})}{F(\text{Hz})} = \frac{3}{2}$$

Since C′ = 48 and we let x = frequency of F in Hz, we can solve for x:

$$\frac{\cancelto{16}{48}\ \text{Hz}}{x} = \frac{\cancelto{1}{3}}{2} \Leftrightarrow x = 32\ \text{Hz}$$

We can now add one more note to our musical scale (Table 5).

Table 5							
C			F	G			C′
24			32	36			48

Our scale now reveals a new interval, the **fourth**: F to C and C′ to G:

$$\frac{\text{F(Hz)}}{\text{C(Hz)}} = \frac{32}{24} = \frac{4}{3} \text{ and } \frac{\text{C}'\text{(Hz)}}{\text{G(Hz)}} = \frac{48}{36} = \frac{4}{3}$$

The matching overtones in a fourth are much more spread out and are therefore it is harder to hear their perichoretic connection.

We can now add D, since it is a fourth below G. Therefore:

$$\frac{\text{G(Hz)}}{\text{D(Hz)}} = \frac{4}{3}$$

Since G = 36 and we let x = frequency of D in Hz, we can solve for x:

$$\frac{\overset{9}{\cancel{36}}\text{ Hz}}{x} = \frac{\overset{1}{\cancel{4}}}{3} \Leftrightarrow x = 27\text{ Hz}$$

We can now add D to our musical scale (Table 6).

Table 6							
C	D		F	G			C′
24	27		32	36			48

We note that A is a fifth above D, E is a fourth below A, and B is a fifth above E. Repeating our algebraic work, we can complete the musical scale (Table 7), a scale unrivaled for its melodies, discovered experimentally by Pythagoras so long ago.

Table 7: The Pythagorean Scale							
C	D	E	F	G	A	B	C′
24	27	30.375	32	36	40.5	45.5625	48

This scale formed the basis for music until the 18$^{\text{th}}$ century, when new studies in harmony altered the frequencies of E, A, and B to generate the **diatonic**[3] **scale**.

Table 8: The Diatonic Scale							
C	D	E	F	G	A	B	C′
24	27	30	32	36	40	45	48

do, re, mi, fa, sol, la, ti, do
C, D, E, F, G, A, B, C′
The musical scale, from the first syllables of a medieval hymn to St. John the Baptist.

[3] Diatonic, from the Greek, literally means "through tone."

In the 18th century, scientist also began to study partial, or fractional, overtones of vibrating strings. We let the fundamental tone be 64 Hz, labeled CC in Table 9. These partial overtones depend upon ever decreasing lengths of the string (think limits), lengths that form the **harmonic sequence** (3rd column of Table 9).

Table 9			
Note	***n***	**Length of String**	**Frequency (Hz)**
CC	1	1	64
C	2	$\frac{1}{2}$	128
G	3	$\frac{1}{3}$	192
C′ (Middle C on the piano)	4	$\frac{1}{4}$	256
E	5	$\frac{1}{5}$	320
G	6	$\frac{1}{6}$	384
B flat	7	$\frac{1}{7}$	448
C″	8	$\frac{1}{8}$	512
D	9	$\frac{1}{9}$	576
E	10	$\frac{1}{10}$	640
F sharp	11	$\frac{1}{11}$	704
G	12	$\frac{1}{12}$	768
A	13	$\frac{1}{13}$	832
B flat	14	$\frac{1}{14}$	896
B natural	15	$\frac{1}{15}$	960
C‴	16	$\frac{1}{16}$	1024

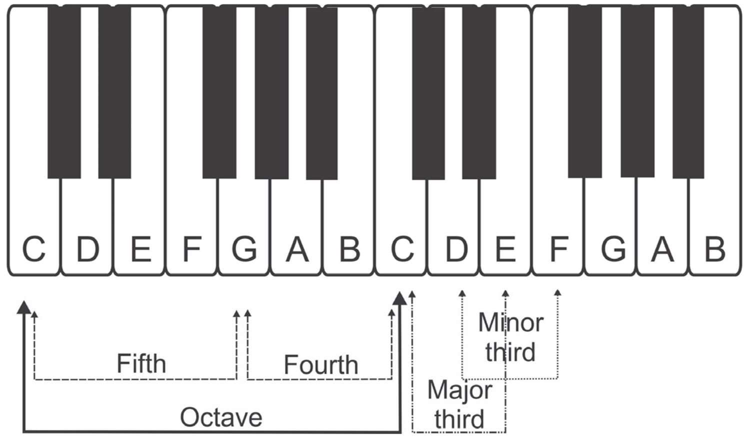

Figure 4. Intervals.

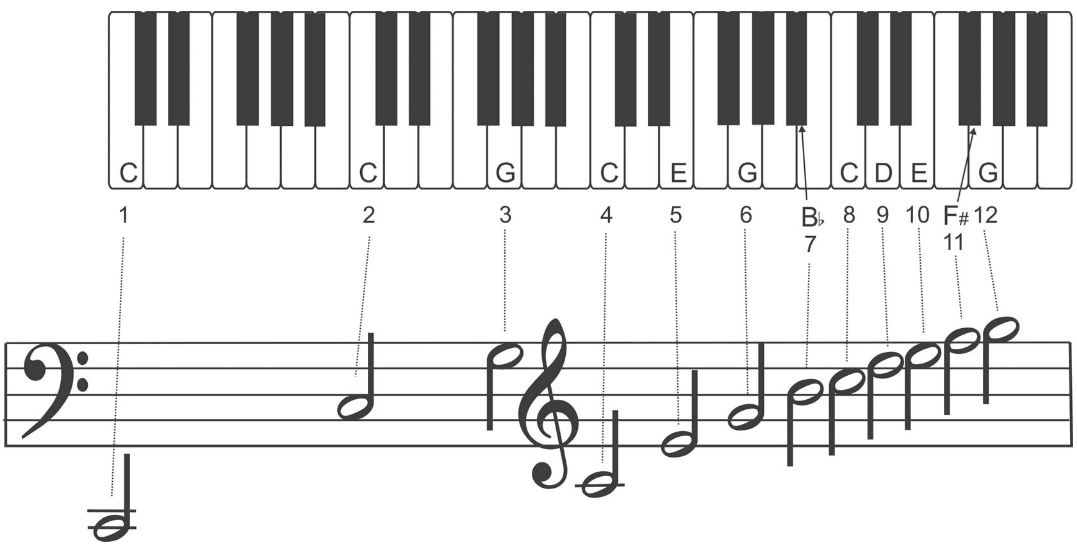

Figure 5. The first twelve partial tones whose fundamental tone is C. Compare with Table 9.

Some of the intervals between notes in this scale have special names, some of which we have already encountered (Table 10).

Table 10	
Interval (*n*)	**Name**
2:1	octave
3:2	fifth
4:3	fourth
5:4	major third
6:5	minor third
9:8	major tone
10:9	minor tone
16:15	semitone

Note also that the harmonic sequence in the 3rd column of Table 9 is the reciprocal of the set of natural numbers:

Set of Natural numbers: {1, 2, 3, 4, 5, 6, ...}

Reciprocal of the set of natural numbers, the harmonic sequence: $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...\right\}$

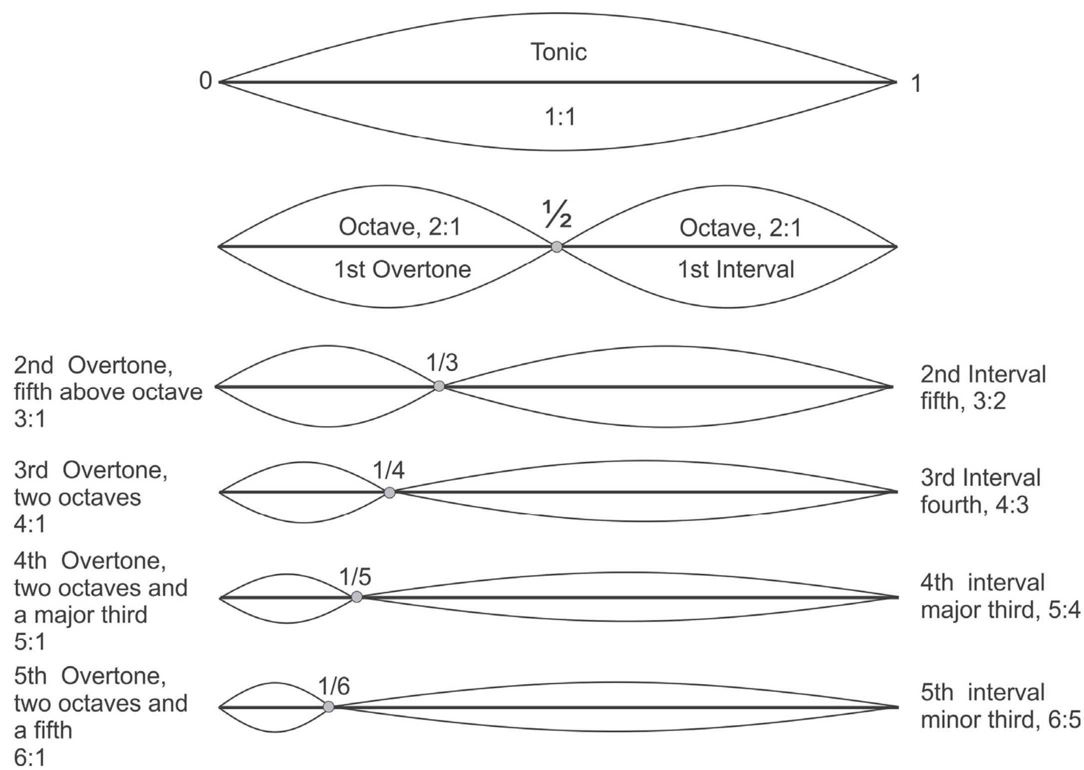

Figure 6. Harmony and stretched strings.

In the sequence of natural numbers, each term of the sequence is the arithmetic mean of the two terms immediately preceding and following it. Recalling that $\mu = \frac{a+b}{2}$, the arithmetic mean of 2 and 4 is 3.

$$\mu = \frac{2+4}{2} = \frac{6}{2} = 3$$

In similar fashion, each term of the sequence, 1, $\frac{1}{2}$, $\frac{1}{3}$, … is the harmonic mean, *h*, of the two terms immediately preceding and following it; i.e., the harmonic mean of $\frac{1}{2}$ and $\frac{1}{4}$ is $\frac{1}{3}$.

How do we compute $\frac{1}{3}$? Given the arithmetic mean, $\mu = \frac{a+b}{2}$, we know that the reciprocal of *a* is $\frac{1}{a}$ and the reciprocal of *b* is $\frac{1}{b}$. Taking the arithmetic mean of these reciprocals, we get:

$$\frac{\frac{1}{a}+\frac{1}{b}}{2}=\frac{\frac{a+b}{ab}}{2}=\frac{a+b}{2ab}$$

THE HARMONIC MEAN

Taking the reciprocal of the arithmetic mean of the reciprocals of a and b (that is a "mouth" full, isn't it?) gives us the harmonic mean. The harmonic mean, h, of any two numbers a and b is:

$$h=\frac{2ab}{a+b}$$

Let's see what happens when $a=\frac{1}{2}$ and $b=\frac{1}{4}$. We calculate h as follows:

$$h=\frac{2\left(\frac{1}{2}\times\frac{1}{4}\right)}{\frac{1}{2}+\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{2}{4}+\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{4}\div\frac{3}{4}=\frac{1}{\cancel{4}_1}\times\frac{\cancel{4}^1}{3}=\frac{1}{3}$$

Is there an algebraic connection between the arithmetic and harmonic means? Given two positive numbers a and b, μ, the arithmetic mean is:

> Music is the pleasure the human mind experiences from counting without being aware that it is counting.
> Gottfried Wilhelm Leibniz (1646-1716).

$$\mu=\frac{a+b}{2}$$

The harmonic mean between any two numbers a and b is:

$$h=\frac{2ab}{a+b}$$

Let's do some Algebra. Follow along:

$$\mu=\frac{a+b}{2}\Leftrightarrow 2\mu=a+b$$

For the harmonic mean, a becomes $\frac{1}{a}$ and b becomes $\frac{1}{b}$:

$$h=\frac{2\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)}{\frac{1}{a}+\frac{1}{b}}\Leftrightarrow h=\frac{\frac{2}{ab}}{\frac{a+b}{ab}}\Leftrightarrow h=\frac{2}{\cancel{ab}_1}\left(\frac{\cancel{ab}^1}{a+b}\right)=\frac{2}{a+b}\Leftrightarrow h(a+b)=2\Leftrightarrow a+b=\frac{2}{h}$$

$2\mu=a+b$ and $a+b=\frac{2}{h}\Rightarrow 2\mu=\frac{2}{h}$

Solving $2\mu = \frac{2}{h}$ for μ, we get:

$$2\mu = \frac{2}{h} \Leftrightarrow \mu = \frac{\cancel{2}^{1}}{\cancel{2}_{1} h} \Leftrightarrow \mu = \frac{1}{h}$$

Solving $2\mu = \frac{2}{h}$ for *h*, we get:

$$2\mu = \frac{2}{h} \Leftrightarrow 2\mu h = 2 \Leftrightarrow h = \frac{\cancel{2}^{1}}{\cancel{2}_{1}\mu} = \frac{1}{\mu}$$

> A peculiar beauty reigns in the realm of mathematics, a beauty which resembles not so much the beauty of art as the beauty of nature and which affects the reflective mind, which has acquired an appreciation of it, very much like the latter.
> E. E. Kummer, *Berliner Monatsberichte* (1867), p. 395.

Note the reciprocal dance, the coinherence, between μ and *h*. It is a consequence of the reciprocal dance between *a* and $\frac{1}{a}$ and between *b* and $\frac{1}{b}$:

1. $\mu = \frac{1}{h}$
2. $h = \frac{1}{\mu}$

> We tend to study all our disciplines in unrelated parallel lines … We have studied exegesis as exegesis, our theology as theology, our philosophy as philosophy; we study something about art as art; we study music as music, without understanding that these are things of man, and the things of man are never unrelated parallel lines.
> Francis Schaeffer (1912-1984), *Francis A. Schaeffer: Trilogy* (1990), p. 212.

The arithmetic mean of *a* and *b* is the reciprocal of the harmonic mean of $\frac{1}{a}$ and $\frac{1}{b}$, and vice versa. If we know the arithmetic mean of two numbers we can easily find the harmonic mean of the reciprocals of those numbers by taking the reciprocal of the arithmetic mean, and vice versa.

If $a = 6$ and $b = 8$, $\mu = 7$ and, therefore, the harmonic mean of $\frac{1}{6}$ and $\frac{1}{8}$ is $\frac{1}{7}$.

THE DANCE OF THE MEANS

Back to musical frequencies (Pythagorean scale), if we let $a = 256$ (the frequency of middle C) and $b = 512$ (the frequency of C′, one octave above C), then $\mu = 384$ (the frequency of G or the fifth). The harmonic mean of 256 and 512 is $341\frac{1}{3}$ (the frequency of F or the fourth). We have a proportion where the ratio of C to F is equal to the ratio of G to C′:

$$\frac{256}{341\frac{1}{3}} = \frac{384}{512}$$ (You multiply to confirm.)

In other words, if we have two frequencies separated by an octave, the arithmetic mean generates the fifth between them (3:2) and the harmonic mean generates the fourth (4:3). The arithmetic mean and the harmonic mean wonderfully interpenetrate fifths and fourths of the Pythagorean number scale. This is perichoresis!

INTERPENETRATION WITH THE WORLD OF OUR EXPERIENCE

In Physics, averages involving rates and ratios are best given using the harmonic mean. Examples include objects working in parallel; e.g., draining water simultaneously with two pumps and the effective resistance of two or more electrical resistors wired in parallel. Space does not allow for detailed explanations of the harmonic mean applied to computer science, hydrology, population genetics, fuel economy, and price/earnings ratios (PE). You can research these connections on your own.

Note the interlacing threads, the perichoretic dance, embedded in these mathematical propositions. Not only are there harmonious relationships between mathematical propositions, there is a corresponding harmony between mathematics and the world, the Triune God's world, the world of our experience.

THE HARMONIC TRIANGLE

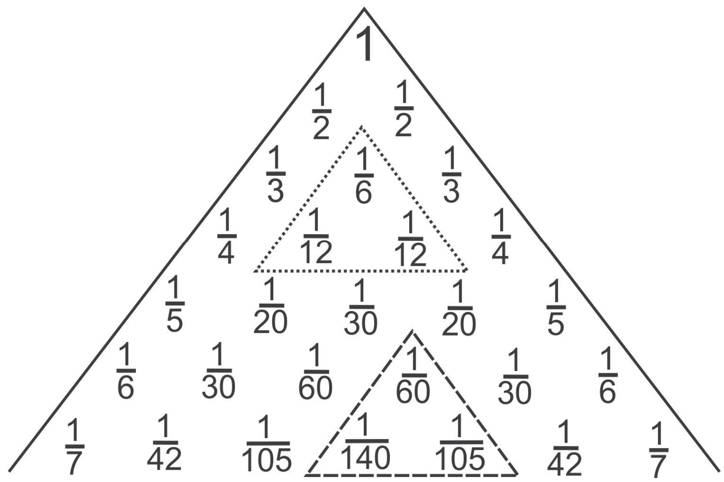

Figure 7. Harmonic Triangle.

We can create a triangle like Pascal's Triangle by incorporating the terms of the harmonic sequence (Figure 7).[4] The **Harmonic Triangle** is also replete with patterns and interconnections.

> The world of ideas which it [mathematics - JN] discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connection of its parts, the infinite hierarchy and absolute evidence of the truths with which mathematical science is concerned, these, and such like, are the surest grounds of its title of human regard, and would remain unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.
>
> James Joseph Sylvester (1814-1897), "A Plea for the Mathematician," *Nature,* Vol. 1(1869-70), p. 262.

[4] The German mathematician Gottfried Wilhelm Leibniz (1646-1716) was the first to generate this triangle so it is often named the Leibniz harmonic triangle in his honor.

EXERCISES

Define the following terms:

1. Harmonic mean
2. Fundamental tone (tonic)
3. Harmonic overtones
4. Frequency (as used in Physics)
5. Octave
6. Musical fifth
7. Musical fourth
8. Pythagorean scale
9. Diatonic scale
10. Harmonic sequence
11. Harmonic triangle
12. Weighted average (See homework exercise below.)

For the ancient Greeks, a mediating proportion defines three relationships between any three unequal numbers a, b, and c, where $a > b > c > 0$. Given a, b, and c, a mediating proportion is when two of their differences ($a - b$ and $b - c$) are to each other in the same relationship as one of these numbers is to itself or to one of the other two numbers. For each of next three questions, (a) solve the proportions for b, the middle term, and (b) describe your result.

13. Possibility 1: $\frac{a-b}{b-c} = \frac{a}{a}$, $\frac{a-b}{b-c} = \frac{b}{b}$, and $\frac{a-b}{b-c} = \frac{c}{c}$ (Solve all three proportions for b.)
14. Possibility 2: $\frac{a-b}{b-c} = \frac{a}{b}$
15. Possibility 3: $\frac{a-b}{b-c} = \frac{a}{c}$

Give an example illustrating the following statements:

16. Three numbers in an arithmetic sequence show an equality of difference, but an inequality of ratio.
17. Three numbers in a geometric sequence show an equality of ratio, but an inequality of difference.

Suppose there are ten females whose average age is 25 and three males whose average age is 35. What is the overall average of the males and females combined? Since there are more females than males, the overall average will be skewed (or weighted) toward 25, not 35. To find the overall average is not simplistic in this case; i.e.,

$$\frac{25+35}{2} = \frac{60}{2} = 30$$

We calculate a **weighted average**[5] as follows:

10 females @ 25 + 3 males @ 35 divided by the total number of males and females or 10 + 3. We compute:

$$\frac{10 \cdot 25 + 3 \cdot 35}{10+3} = \frac{355}{13} \approx 27.31$$

[5] Weighted average is an average resulting from multiplying each data value by a factor reflecting its importance.

Answer the following questions:

18. 80 people earn an average of $40,000 per annum (year) for 30 years and 60 people earn an average of $30,000 per annum for 20 years. What is the overall average, rounded to the nearest dollar?
19. A teacher gives three exams during the school year and weights them in this manner: 15% for Exam #1, 25% for Exam #2, and 60% for Exam #3. A student scores 75% for Exam #1, 85% for Exam #2, and 95% for Exam #3.
    (a) What is the student's weighted average percentage mark for these three exams?
    (b) Compare this mark with the average of the three scores

Answer the following questions:

20. In Table 7, show how the exact frequency of A, E, and B is derived.
21. Demonstrate that the harmonic mean of:
    (a) 24 and 48 is 32.
    (b) the harmonic mean of 256 and 512 is $341\frac{1}{3}$.
22. Establish a proportion that connects the arithmetic and harmonic means of 24 and 48.
23. Show that the harmonic sequence is denumerable.
24. What is the ratio of (a) 1/2 to 1/3? (b) 1/3 to 1/4? (c) 1/4 to 1/5? (d) 1/5 to 1/6?
25. What is the relationship of the ratios in Question 24 to musical intervals (Table 10)?
26. Calculate the:
    (a) arithmetic mean of 7 and 9.
    (b) harmonic mean of 7 and 9.
27. Calculate the:
    (a) arithmetic mean of 1/7 and 1/9.
    (b) harmonic mean of 1/7 and 1/9.
28. What relationship do you see in your answers to Questions 26 and 27?

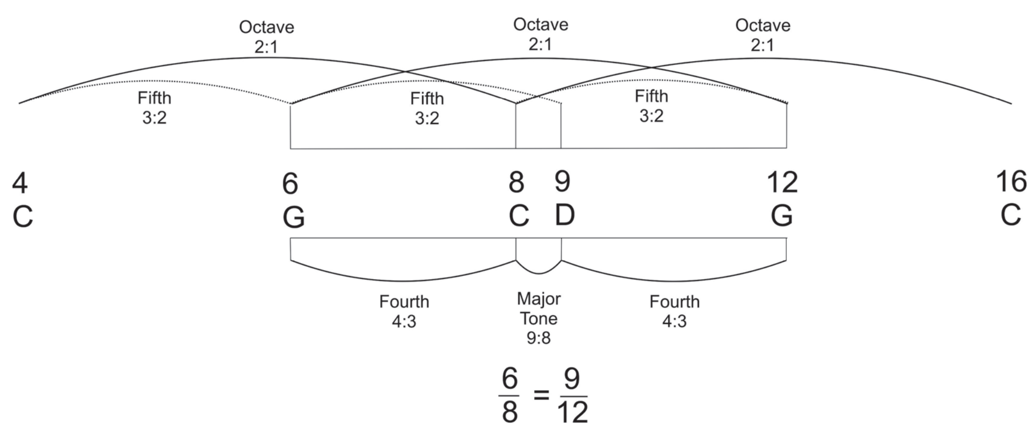

Figure 8. Octave, fifth, fourth, and major tone.

In Table 11 and Figure 8, we see that the arithmetic and harmonic means work out a perfection through an interchange of differences in a play of a variance of proportional relationships. The respective frequencies of the sequence G, C, D, and G represent a fundamental tone, the fourth, the fifth, and the octave. Note the relationship between string length and frequency and answer these questions.

Table 11: The Musical Proportion

	Note	Fourth (Harmonic Mean)	Fifth (Arithmetic Mean)	Octave
Frequency	1	$\frac{4}{3}$	$\frac{3}{2}$	2
	6 (G)	8 (C)	9 (D)	12 (G)
String Length	12 (G)	9 (C)	8 (D)	6 (G)
	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$
	Note	**Fourth (Arithmetic Mean)**	**Fifth (Harmonic Mean)**	**Octave**

29. Confirm the four harmonic and the four arithmetic means.
30. Identify the reciprocal ratios where the harmonic and arithmetic means are fractions.
31. Identify the crossing of functional positions between the arithmetic and harmonic mean.
32. (a) Identify the frequency and string length proportions where the harmonic and arithmetic means are fractions.
(b) What do these proportions tell you about the relationship between the fundamental tone, the fourth, the fifth, and the octave?

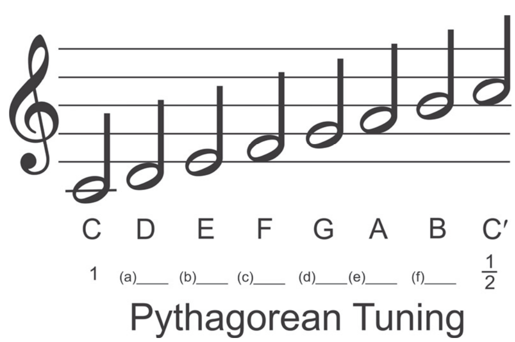

Figure 9. Pythagorean musical scale.

Answer the following questions:

33. In Figure 9, if the length of a vibrating spring representing C on the musical scale is 1 unit (the first term of the harmonic sequence), then the length of the string one octave higher, C′, is 1/2 unit (the second term of the harmonic sequence). Calculate the parts of the whole string for the notes (a) D, (b) E, (c) F, (d) G, (e) A, and (f) B. (Hint: the ratios will be in descending order and between 1 and 1/2.)

34. The number 0.01001000100001000001000001 … is an irrational number with only two digits. What is the pattern in this number?
35. Can you create another irrational number with only two digits?
36. What is x in the equation $\lim_{n \to \infty} \frac{1}{n} = x$?

Answer the following questions are about the Harmonic Triangle (Figure 10):

37. In the second row, how is 1/6 derived?
38. In the third row, how is 1/12 derived?
39. In the fourth row, how is:
    (a) 1/20 calculated?
    (b) 1/30 calculated?
40. What is the rule for finding the numbers in the Harmonic Triangle?
41. In the Harmonic Triangle encompassed by three dotted line segments, what is the relationship between 1/6, 1/12, and 1/12?
42. In the triangle encompassed by three dashed line segments, what is the relationship between 1/60, 1/140, and 1/105?
43. Develop a formula for finding the sum of the denominators for each row. (Hint: Let the apex of the triangle, 1, be row 1 and think powers of 2.)

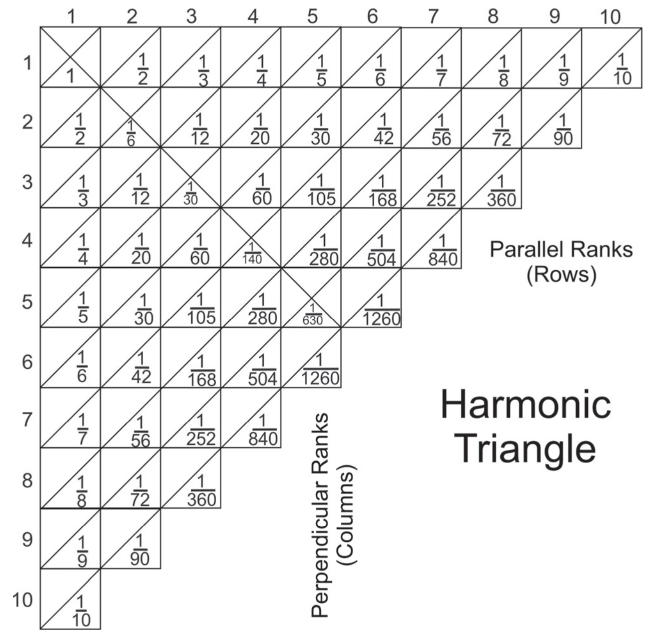

Figure 10. A different view of the Harmonic Triangle.

Use Figure 10 to answer the following questions:

44. Using a calculator, find the sum of the fractions in:
    (a) row 2 rounded to the nearest one [Note: In a homework exercise in Lesson 8.5, we have found the infinite sum of $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots$.]
    (b) row 3 rounded to the nearest tenth.
    (c) row 4 rounded to the nearest tenth. (Assume the existence of a periodic decimal.)
    (d) row 5 rounded to the nearest tenth. (Note: these sums are approximations of the infinite sums comprising each row of the triangle.)
    (e) What do you notice?
45. (a) Determine a formula for finding the n^{th} number in the second row of the Harmonic Triangle. (Note: the formula must be in terms of n.)
    (b) Determine a formula for finding the n^{th} triangular number (Lesson 6.1). (Again, the formula must be in terms of n.)
    (c) Compare the formula in your answer in (a) with the formula in your answer to (b). Describe what you see. (Hint: Think reciprocal.)
46. Use your results from the previous two questions to determine the infinite sum of the reciprocals of the triangular numbers; i.e., $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots$ [Note: The Dutch mathematician Christiaan Huygens (1629-1694) asked Leibniz to find this sum to help solve a problem involving the computation of probabilities for certain games of chance. Isn't it amazing how the harmony of music is involved in probability theory? Perichoresis in action!]

What operation is performed on the equation of the left to generate the equivalent equation on the right?

47. $x \cdot \frac{3}{2} = C' \Leftrightarrow x = C' \cdot \frac{2}{3}$

48. $\mu = \frac{a+b}{2} \Leftrightarrow 2\mu = a + b$

49. $b(a+b) = 2 \Leftrightarrow a + b = \frac{2}{b}$

50. $2\mu = \frac{2}{b} \Leftrightarrow \mu = \frac{1}{b}$

51. $2\mu = \frac{2}{b} \Leftrightarrow b = \frac{1}{\mu}$

> Without the belief that it is possible to grasp the reality with our theoretical constructions, without the belief in the inner harmony of our world, there could be no science. This belief is and always will remain the fundamental motive for all scientific creation.
>
> Albert Einstein and Leopold Infield, *The Evolution of Physics: The Growth of Ideas from the Early Concepts to Relativity and Quanta* (1938), p. 312.

Field Project:

52. (a) Gather a significant number (more than 500) of marbles, jelly beans, or coins and count them exactly.
(b) Place all of them in a clear jar.
(c) If you can, find at least 100 people and ask each one to guess the number of objects in the jar and record each guess.
(d) Take the arithmetic mean of the guesses. What do you notice? Why?

53. The search engine company Google can predict certain events (e.g., a flu epidemic) by creating a database of words the people search for (e.g., medicine, flu, fever, causes of headaches, etc.). Based on your answer to Question 52, why is such a prediction possible?

> May not music be described as the mathematics of the sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music: music the dream, mathematics the working life.
>
> James Joseph Sylvester (1814-1897).