Euler's Crown Jewel

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Complex numbers (of the form $a + bi$ where $i = \sqrt{-1}$) are indispensable, not only in the manifold beauties of fractal geometry, but in understanding the theory of alternating electric current, specifically in the study of the impedance, inductive and capacitive reactance, and the phase angle. In 1893, Charles Steinmetz (1865-1923), an engineer with General Electric found that complex numbers were exactly what he needed to describe this current (it permits electrical power to be transmitted long distances at minimal cost). Today, every electrical engineer uses $\sqrt{-1}$, though the symbol used to represent it is not i but j (the symbol i is reserved in electrical engineering texts to represent current).

This essay is extracted from a lesson from the forthcoming textbook *The Dance to Infinity*.

This type of application is common in mathematics. Often the concepts that a mathematician "creates" are applied years later in a field that would have never crossed the mind of the mathematician.[1] Alternating current had not been discovered when Gauss published his findings of complex numbers. Seven years later (in 1822), Michael Faraday (1791-1867), an English physicist and chemist (and dedicated Christian), wrote in his notebook the idea about converting magnetism into electricity. He achieved this goal in 1831 when he used this conversion to generate alternating electric current.

Imaginary numbers also come into play in the study of modern quantum physics. In 1963, Eugene Wigner (1902-1995) won the Nobel Prize in physics for his research in quantum mechanics. He said this about $\sqrt{-1}$:

Eugene Wigner. Source: Public Domain

"… the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics."[2]

Given this fact, he responds with this amazing remark:

"It is difficult to avoid the impression that a miracle confronts us here."[3]

And again, he ponders:

"The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and … there is no rational explanation for it."[4]

The miracle that Wigner cannot explain is the fact that the mathematics of complex numbers fits the theory of quantum mechanics like a glove fits a hand. Here is another example of the truth that this coherence exists because the Biblical God created the physical world with its mathematical properties and the human mind with its mathematical capabilities. Because Wigner denies this God, he is faced with miracles and mystery. The Biblical Christian sees this connection and worships his Triune Creator with awe-inspiring understanding, unlike Wigner, who is only faced with a self-imposed dumbfounded mystery.

[1] In this context, it is important that we understand the role of the guiding hand of God (often called Providence) in the development of mathematical theories.

[2] Eugene Wigner, *Symmetries and Reflections: Scientific Essays* (Cambridge and London: The MIT Press, 1970), p. 229.

[3] *Ibid.*

[4] *Ibid.*, p. 223.

There is one more astounding wonder about imaginary numbers that I would like to explore in this essay. The reasoning I am about to give you was first done by Leonhard Euler (1707-1783), the man of whom it has been said, "As he breathes, he calculates." If you pay attention carefully, then your reward will be great. I also hope that you will appreciate the ingenious mind that God gave to Euler for not many people in God's providence are gifted as such.


Leonhard Euler. Source: Public Domain

Euler's inexplicable formula involves i, π, and e, three amazing numbers.[5] Early in the history of the study of logarithms, e was defined as follows:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Euler used this definition to develop a power series for e.[6] He first recognized that he could apply Pascal's triangle (the binomial formula) to this definition. He began with the combination formula for deriving the terms of a binomial expansion:

$${}_nC_j = \binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

Let's review how this formula calculates the 6th row of Pascal's triangle:

$${}_6C_0 = \binom{6}{0} = \frac{6!}{6!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} = 1$$

$${}_6C_1 = \binom{6}{1} = \frac{6!}{5!1!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 6$$

$${}_6C_2 = \binom{6}{2} = \frac{6!}{4!2!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot2\cdot1} = 15$$

$${}_6C_3 = \binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot3\cdot2\cdot1} = 20$$

$${}_6C_4 = \binom{6}{4} = \frac{6!}{2!4!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot4\cdot3\cdot2\cdot1} = 15$$

$${}_6C_5 = \binom{6}{5} = \frac{6!}{1!5!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 6$$

$${}_6C_6 = \binom{6}{6} = \frac{6!}{6!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} = 1$$

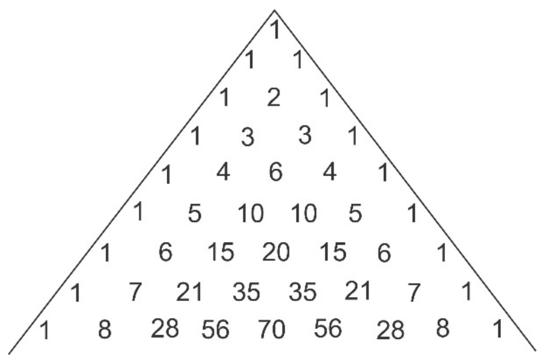

Figure 1: Pascal's Triangle

The 6th row generates the coefficients of 1, 6, 15, 20, 15, 6, and 1. Note that we can rewrite $n!$ as follows:

$$n! = 1\cdot2\cdot3\cdot\ldots\cdot n$$

We can also rewrite $(n - j)!$ as follows:

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[5] As we have noted, $i = \sqrt{-1}$, and both e and π are transcendental numbers. Technically, a transcendental number is a real or complex number that is *not* algebraic; i.e., not a solution of a non-zero polynomial equation having with rational coefficients. $e \approx 2.71828$ and $\pi \approx 3.14159$.

[6] Technically, a power series (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

$$(n-j)! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-j)$$

Now we can rewrite the combination formula as follows:

$${}_nC_j = \binom{n}{j} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-j)\cdot j!}$$

Note that all the factors from 1 to $(n-j)$ in the numerator cancel with those in the denominator, leaving only these factors in the numerator:

$$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-j+1)$$

Now, we can write the combination formula as:

$${}_nC_j = \binom{n}{j} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-j+1)}{j!}$$

Let's apply this formula to calculate the 6th row of Pascal's triangle (recall that 0! = 1):

$${}_6C_0 = \binom{6}{0} = \frac{0!}{0!} = \frac{1}{1} = 1$$

$${}_6C_1 = \binom{6}{1} = \frac{6}{1!} = 6$$

$${}_6C_2 = \binom{6}{2} = \frac{6\cdot 5}{2\cdot 1} = 15$$

$${}_6C_3 = \binom{6}{3} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$$

$${}_6C_4 = \binom{6}{4} = \frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2\cdot 1} = 15$$

$${}_6C_5 = \binom{6}{5} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 6$$

$${}_6C_6 = \binom{6}{6} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 1$$

With this equation, we can now apply the binomial formula to $\left(1+\frac{1}{n}\right)^n$ as follows:

$$\text{(1.1)}\quad \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n\cdot(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 + \frac{n\cdot(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3 + \ldots + \left(\frac{1}{n}\right)^n$$

Note that in expanding the third term we get:

$$\frac{n\cdot(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 = \frac{n\cdot(n-1)}{2!}\frac{1}{n^2} = \frac{n-1}{2!n}$$

We can rewrite $\frac{n-1}{n}$ as follows (note: $\frac{9}{10} = 1 - \frac{1}{10}$):

$$\frac{n-1}{n} = \frac{n}{n} - \frac{1}{n} = 1 - \frac{1}{n}$$

Therefore, we get:

$$\frac{n\cdot(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 = \frac{\left(1-\frac{1}{n}\right)}{2!}$$

With these algebraic manipulations in mind, we get:

$$(1.2)\qquad \left(1+\frac{1}{n}\right)^n = 1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\ldots+\frac{1}{n^n}$$

When a baseball player works the count, gets his pitch, and drives a single into the outfield, seasoned baseball experts usually remark, "That was a fine piece of hitting." From (1.1) to (1.2) we have followed Euler's "fine piece of thinking." But, there are more of the "quintessential thoughts" of Euler to come.

Since we are looking for the *limit* of $\left(1+\frac{1}{n}\right)^n$ as $n\to\infty$, we must let n increase without bound. Our expansion will have more and more terms. At the same time, the expression within each pair of parentheses will tend to 1 since the limits of $\frac{1}{n},\frac{2}{n},\ldots$ as $n\to\infty$ are all 0. From this, Euler produced the following:

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

This series is a beautiful and patterned series of e. He then replaced $\frac{1}{n}$ by $\frac{x}{n}$ and got:

$$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

We can show that this series converges for all real values of x; in fact, the rapidly increasing denominators cause the series to converge very quickly. It is from this series that numerical values of e^x are calculated (this series is programmed into modern scientific calculators when the e^x key is punched).

Euler continued to experiment with the power series for e^x. He decided to see what happened to this series if he replaced x with the *imaginary* expression ix where, of course, $i=\sqrt{-1}$. This substitution took some nerve on Euler's part since e^x had always represented a real number. Now Euler was going to see what would happen to the expression e^{ix}. This is what he got:

$$e^{ix}=1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}\ldots$$

Knowing the properties of i, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$, etc., he now got:

$$e^{ix}=1+ix-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}+\frac{ix^5}{5!}-\frac{x^6}{6!}\ldots$$

Now Euler did what today's mathematicians consider unthinkable. He changed the order of the terms collecting all the real terms separately from the imaginary terms. With an infinite series, this could generate trouble. Instead of converging to a limit, it may diverge. Factoring out i from the imaginary terms, he now got:

$$e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\ldots\right)$$

In Euler's time, the power series of sin x and cos x (x in radians) was well known.[7] They are defined as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Given these definitions, *Euler made a grand substitution*. He got:

(1.3) $$e^{ix} = \cos x + i \sin x$$

This equation expresses a surprising link between the exponential function (albeit raised to an imaginary power multiplied by the variable x) and basic trigonometry ratios.[8] Euler now replaced ix by $-ix$ and using the trigonometric identities cos ($-x$) = cos x and sin ($-x$) = -sin x, he rewrote equation (1.3) as follows:

(1.4) $$e^{-ix} = \cos x - i \sin x$$

Next, he added equation (1.3) and equation (1.4). He got:

(1.5) $$e^{ix} + e^{-ix} = \cos x + i \sin x + \cos x - i \sin x = 2 \cos x$$

Combining like terms, he got:

(1.6) $$e^{ix} + e^{-ix} = 2 \cos x$$

Dividing both sides of equation (1.6) by 2, he got:

(1.7) $$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Then, he subtracted equation (1.4) from equation (1.3) and got:

(1.8) $$e^{ix} - e^{-ix} = \cos x + i \sin x - \cos x + i \sin x$$

Combining like terms, he got:

(1.9) $$e^{ix} - e^{-ix} = 2i \sin x$$

Dividing both sides of equation (1.9) by $2i$, he got:

(1.10) $$\sin x = \frac{e^{ix} + e^{-ix}}{2i}$$

Every step that Euler has taken so far (and some steps were giant intuitive leaps) has been confirmed through the rigors of analysis in the 19th and 20th centuries. Euler, like many of his time, was a pioneer. He blazed many trails and left it for others to confirm his steps.

There is more to come. Since x is in radians, then Euler let $x = \pi$ (180°). Since cos π= -1, from equation (1.7), Euler got:

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[7] These are known as the Taylor series named in honor of the English mathematician Brook Taylor (1685-1731).
[8] For another way to derive this equation, see www.biblicalchristianworldview.net/Mathematical-Circles/stunningConnection.pdf

(1.11) $\cos\pi = -1 = \dfrac{e^{i\pi} + e^{-i\pi}}{2}$

Multiplying both sides of equation (1.11) by 2, he got:

(1.12) $-2 = e^{i\pi} + e^{-i\pi}$

Since sin π = 0, from equation (1.10), Euler got:

(1.13) $\sin\pi = 0 = \dfrac{e^{i\pi} - e^{-i\pi}}{2i}$

Multiplying both sides of equation (1.13) by 2i, he got:

(1.14) $0 = e^{ix} - e^{-i\pi}$

Adding $e^{-i\pi}$ to both sides of equation (1.14), he got:

(1.15) $e^{-i\pi} = e^{i\pi}$

Now, Euler substituted equation (1.15) into equation (1.12) and got:

(1.16) $-2 = 2e^{i\pi}$

Dividing both sides of equation (1.16) by 2, he got:

(1.17) $e^{i\pi} = -1$

Adding 1 to both sides of equation (1.17), he derived his *crown jewel:*

(1.18) $e^{i\pi} + 1 = 0$

Mathematicians have described equation (1.18) using words like "remarkable," "beautiful," "a revelation," "absolutely paradoxical," "certainly true," "incomprehensible," and "cannot be explained in words." Hats off to Euler!

Here we have an equation that *connects* the five most important constants of mathematics (0, 1, *i*, π, *e*) and three of the most important mathematical operations (addition, multiplication, and exponentiation). The five constants connect four major branches of mathematics: arithmetic (0 and 1), algebra (*i*), geometry (π), and analysis (*e*).

> There is a famous formula – perhaps the most compact and famous of all formulas – developed by Euler from a discovery of De Moivre: $e^{i\pi} + 1 = 0$ …. It appeals equally to the mystic, the scientist, the philosopher, the mathematician.
>
> Edward Kasner & James Newman, *Mathematics and the Imagination* (1940).

This formula is so tantalizing that some believe it appeals to the mystic, the scientist, and the philosopher. For the Biblical Christian, this formula serves as another resounding echo of the voice of the ultimate One and the Many, the Triune Author and Sustainer of all rationality including this striking connection, the onto-relational being who is the ground of the astonishing and wondrous unity and diversity in mathematics.