RECURSION AND THE FIBONACCI SEQUENCE

BY JAMES D. NICKEL

In an *arithmetic sequence*, a common "difference" separates each term in the sequence. Using function notation, every arithmetic sequence is a linear function of the form y = f(x) = ax + b (where the domain is the positive integers and *a* is the common difference). In a *geometric sequence*, you calculate each successive term by *multiplying by the same number*. Using function notation, every geometric sequence is an exponential function of the form $y = f(x) = ar^{x-1}$ (where the domain again is the positive integers and *r* is the common ratio or multiplier). In a *power sequence*, we calculate each successive term by raising consecutive positive integers to the same power.

The next sequence of importance is the *recursive* sequence. Arithmetic and geometric sequences can be understood using recursion methods. Recursion is a method[1] where, based upon a starting term, you repeatedly applying the same arithmetic operation to each term to arrive at the subsequent term. In *recursion, you must know the value of the term immediately before the term you are trying to find.*

A recursive formula always has two parts. First, we choose the starting value or t_1 (the first term or "t sub 1"). Second, a recursion formula for t_n ("t sub n") as a function of t_{n-1} ("t sub n – 1"), the term preceding t_n.

Here is a simple example. Let $t_1 = 5$. Our formula is $t_n = 5t_{n-1}$. This is all we need to generate the sequence: 5, 25, 625, 3125, … In this example, we have created a geometric sequence (common ratio or multiplier is 5).

Term	t_1	t_2	t_3	t_4
Value	5	25	625	3125

Next, we let $t_1 = 5$ and our formula $t_n = t_{n-1} + 5$. Our sequence: 5, 10, 15, 20, … (an arithmetic sequence!).

Term	t_1	t_2	t_3	t_4
Value	5	10	15	20

One of the more famous sequences that we can understand by recursive methods is the Fibonacci sequence, a sequence related to the growth of populations. In 1788, European settlers brought the rabbit to the continent of Australia. Now, the wild rabbit population is over 200,000,000 (compared to the human population of Australia of 20,000,000). Rabbits have no natural predators and, because of their rapid population growth, they do millions of dollars of damage to Australian agriculture every year.


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It was the study of the growth of rabbit populations that led a 13th century European mathematician to the discovery of the Fibonacci sequence.[2] He was Leonardo of Pisa (1170-1240), also known as Leonardo Fibonacci (son of Bonaccio, his father). Here is the sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

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[1] We also call these methods *algorithms*, a set of rules for solving a problem in a finite number of steps. The word is Arabic coming from algorism, a Latinization of the Arab mathematician al-Khwarizmi (ca. 780-ca. 850), in important figure in the development of algebra.

[2] The Fibonacci sequence was known in ancient India, where it was applied to the metrical (poetry)sciences (prosody), long before it was known in Europe. Developments have been attributed to Pingala (200 BC), Virahanka (6th century AD), Gopāla (ca. 1135 AD), and Hemachandra (ca. 1150 AD).

What is happening? The first two terms are 1, and each successive term is the sum of the preceding pair of terms. We let F_n represent the n[th] term of this Fibonacci sequence. In this sequence, we must know the value of the first *two* terms.

$F_1 = 1$ and $F_2 = 1$. What is the formula? $F_{n+1} = F_n + F_{n-1}$. To "work" the sequence, we must start at n = 2. Note this table:

Term	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Value	1	1	2	3	5	8	13	21	34

We can better see what is going on by studying this table:

n	F_{n-1}	F_n	F_{n+1}
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8
6	5	8	13

Notice how the terms of the Fibonacci sequence unfold in columns 2, 3, and 4 (labeled F_{n-1}, F_n, and F_{n+1} respectively).

Fibonacci, a traveling merchant, visited Arabia. There he discovered the Hindu-Arabic system of numeration (decimal/positional notation). In 1202, he wrote *Liber abaci* (meaning "Book of calculation"), and by it, introduced this system of numbers to Europe. This book is a collection of business mathematics, linear and quadratic equations, including square roots and cube roots, and other topics. He also said,

> "These are the nine figures of the Indians 9 8 7 6 5 4 3 2 1. With these nine figures, and with the symbol 0, which in Arabic is called *zephirum* [means 'empty'–JN], any number can be written, as will be demonstrated below."[3]

In this book, he wrote about the results of his study of rabbit populations. He wanted to know the number of pairs of rabbits produced each year if one begins with a single pair that matures during the first month and then produces another pair of rabbits after that (of course, he is assuming that no rabbits die in this process!). The Fibonacci sequence answers that question. How?

Suppose there is one pair of rabbits in the month of January that breed a second pair in the month of February. Thereafter, these produce another pair monthly; i.e., each pair of rabbits produces another pair in the second month following birth and thereafter one pair per month. How many pairs will there be after one year?

To solve this problem, we need to tabulate four columns:

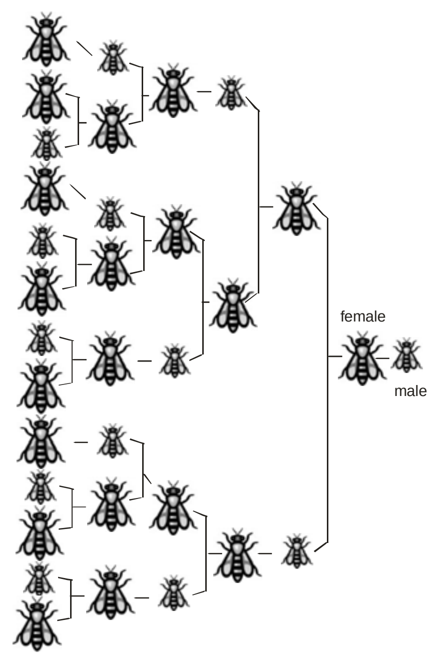
1. The number of pairs of breeding rabbits at the beginning of a given month.
2. The number of pairs of non-breeding rabbits at the beginning of the month.
3. The number of pairs of rabbits bred during the month.
4. The number of pairs of rabbits living at the end of the month.

[3] Cited in Alfred S. Posamentier, *Math Wonders* (Alexandria, VA: Association for Supervision and Curriculum Development, 2003), p. 32.

Month	*1* *Breeding*	*2* *Non-breeding*	*3* *Bred*	*4* *Total*
January	0	1	0	1
February	1	0	1	2
March	1	1	1	3
April	2	1	2	5
May	3	2	3	8
June	5	3	5	13
July	8	5	8	21
August	13	8	13	34
September	21	13	21	55
October	34	21	34	89
November	55	34	55	144
December	89	55	89	233

Again, note how the terms of the Fibonacci sequence unfold in the columns labeled 1, 2, 3, and 4.

Since the Fibonacci sequence connects to the growth of certain populations, we frequently find it in the physical creation, especially of the biological nature. Let's see how it reveals the genetics of the honeybee (*haplodiploid reproduction*). The drone (male bee) hatches from an egg that has not been fertilized (it has only one parent, its mother). The fertilized egg produces only a female, either a queen bee or a worker bee (it has two parents, a father and a mother).



If we look at just one generation, a drone has one parent, a mother. If we look at the second generation of ancestors, the drone has two grandparents: the mother and father of his mother. At the third generation, there are three great grandparents: the mother and father of his grandmother, and the mother (no father) of his grandfather. If you figure out the parentage of his great grandparents, you will find that there are five great, great grandparents. Taking the drone himself as the starting generation, the number of bees in each generation follows the Fibonacci sequence:

Generation	*Number of Bees*
1	1 drone
2	1 mother
3	2 grandparents
4	3 great grandparents
5	5 great, great grandparents
6	8 great, great, great grandparents
7	13 great, great, great, great grandparents

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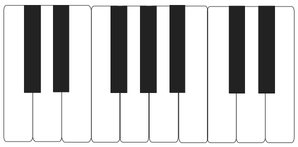
Man, made in the image of God, has used the properties of this sequence in his art, architecture, and music. For example, the piano keys shown are for one *octave* (a series of a group of 8) of the *chromatic scale* on



the piano. The number of black keys in the octave is 5 (these keys used to be called the *pentatonic scale* where "penta" is Greek for 5), which is the fifth term of the Fibonacci sequence.

The number of white keys in the octave is 8 (these keys are called the "major" scale), which is the sixth term of the sequence. The total number of keys is 13 (the seventh term). Note also that the black keys are structured in sets of 2 and 3 (also Fibonacci numbers).

Fibonacci numbers are also revealed in the spiral arrangement of petals, pinecones, and pineapples (check this out in a grocery store or with some pinecones). In the pinecone spiral, there are 5 spirals one way and 8 the other. In pineapple spirals, the pair one way and the other can be 5 and 8, or 8 and 13. In the sunflower spirals, the combination can be 8 and 13, 21 and 34, 34 and 55, 55 and 89, and 89 and 144. Truly, the Fibonacci sequence is ubiquitous in God's creation!

Source: iStockphoto

We can compare the growth of power sequences (the sequence of squares and the sequence of cubes) and Fibonacci numbers (much more "explosive") by observing this table:

Number	Square	Cube	Fibonacci Numbers	
1	1	1	F_1	1
2	4	8	F_2	1
3	9	27	F_3	2
4	16	64	F_4	3
5	25	125	F_5	5
6	36	216	F_6	8
7	49	343	F_7	13
8	64	512	F_8	21
9	81	729	F_9	34
10	100	1000	F_{10}	55
11	121	1331	F_{11}	89
12	144	1728	F_{12}	144
13	169	2197	F_{13}	233
14	196	2744	F_{14}	377
15	225	3375	F_{15}	610
16	256	4096	F_{16}	987
17	289	4913	F_{17}	1597
18	324	5832	F_{18}	2584
19	361	6859	F_{19}	4181
20	400	8000	F_{20}	6765
21	441	9,261	F_{21}	10,946
22	484	10,648	F_{22}	17,711
23	529	12,167	F_{23}	28,657
24	576	13,824	F_{24}	46,368
25	625	15,625	F_{25}	75,025

Number	Square	Cube	Fibonacci Numbers	
26	676	17,576	F_{26}	121,393
27	729	19,683	F_{27}	196,418
28	784	21,952	F_{28}	317,811
29	841	24,389	F_{29}	514,229
30	900	27,000	F_{30}	832,040

Let's conclude our brief study of the Fibonacci sequence by revealing it in a number trick. I want you to create a table (10 rows by 2 columns). Pick any two numbers where each number is a two-digit number. In my example, I will use 15 and 18 as my starting number. Make sure you choose two numbers different from these.

Write these two numbers in rows 1 and 2. Add them (e.g., 15 + 18 = 33) and write your sum in row 3 (you are engaging the Fibonacci recursion algorithm). Next, add your sum to the second number you chose (e.g, 18 + 33 = 51) and write this sum in row 4. Continue this process until you reach row 10.

Can you determine a fast way to determine the sum of all the numbers in the second column?

1.	15
2.	18
3.	33
4.	51
5.	84
6.	135
7.	219
8.	354
9.	573
10.	927
Sum	?

We invoke algebra to help us find this sum. Let *a* and *b* represent the two numbers you chose. The following table explains what you are doing in terms of *a* and *b*. In review, note a + a = 2a and b + 2b = 3b

1.	a
2.	b
3.	a + b
4.	a + 2b
5.	2a + 3b
6.	3a + 5b
7.	5a + 8b
8.	8a + 13b
9.	13a + 21b
10.	21a + 34b
Sum	55a + 88b

Now, sum up all the *a's* and *b's* (in algebra, we are summing "like terms") in this table. You should get 55a + 88b (this represents the sum of all the numbers in the second column). Note the appearance of the Fibonacci numbers as coefficients of *a* and *b*. Compare this sum with row 7; i.e., 55a + 88b with 5a + 8b. The sum is 11 times the value of row 7! Using the distributive rule, 11(5a + 8b) = 55a + 88b. To find the sum, all you need to do is multiply row 7 by 11!

In my example where I selected 15 and 18 as my starting numbers), 219 × 11 = (use the distributive rule again) = 219(10 + 1) = 2190 + 219 = 2409! Such is the power of algebra!