

A Simple Proof that e is Irrational

By James D. Nickel

Two of the most beautiful and significant numbers in mathematics are π and e . π (approximately equal to 3.14159) represents the ratio of the circumference of a circle to its diameter. e (approximately equal to 2.71828) is the base of the natural logarithms. Both π and e are identified by mathematicians as transcendental numbers. Transcendental numbers are numbers that cannot be expressed as the root (i.e., solution) of any algebraic equation with rational coefficients. This means that π cannot *exactly* satisfy equations as follows:

$$\begin{aligned}\pi^2 &= 5 \\ 2\pi^4 - 18\pi^2 + 4\pi - 6 &= 0\end{aligned}$$

Note that these equations involve integers (technically, integers are rational numbers) with powers of π . You can replace π with e and come to the same conclusion: e cannot *exactly* satisfy these equations.

Although the numbers π and e are not solutions of algebraic equations, these numbers can be expressed as infinite continued fractions or as the limit of an infinite series. Note the beautiful patterns that unfold in the following examples:

$$\pi = 3 + \frac{1^2}{6 + \frac{3^2}{6 + \frac{5^2}{6 + \frac{7^2}{6 + \frac{9^2}{6 + \frac{11^2}{6 + \dots}}}}}} \quad (\text{Infinite continued fraction})$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \quad (\text{Infinite series})$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \quad (\text{Infinite series})^1$$

In 1882, German mathematician Ferdinand von Lindemann (1852-1939) proved that π is transcendental, putting an end to nearly *2500 years of conjecture*. By his proof he showed that π *transcends* the power of algebra to display it in its totality; i.e., π cannot be expressed in any *finite* series of arithmetical or algebraic operations. This “transcending of the finite” is why the word *transcendental* is used to describe a number like π .

The decimal expansion of π and e never end, nor has anyone detected an orderly pattern in their arrangement.² Hence, these numbers are also irrational; i.e., they cannot be written as the ratio of two integers.³

The mathematician's patterns like those of the painter's or the poet's must be beautiful. The ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test. There is no permanent place in the world for ugly mathematics.

G. H. Hardy (1877-1947)

¹ In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$.

² Note, amazingly, that the frequency rate of the decimal digits in the decimal expansion of π is equal (i.e., the digit 0 occurs 10%, 1 occurs 10%, 2 occurs 10%, etc.).

³ Other irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.

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With computers, we have been able to “estimate” π to nearly one *trillion* digits. Using a fixed-size font, we could not write the decimal expansion of π on a piece of paper *stretching from one end of the universe to the other!*

The proof that π is an irrational number is a challenge to follow, but it can be done. Proving that e is an irrational number is slightly easier (of course, “easy” is in the eyes of the beholder!).

To ready ourselves for the proof that e is irrational, let’s first prove that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$. To do this, we must first prove that $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. We note the following:

$$\begin{aligned}1 &= \frac{1}{2} + \frac{1}{2} \\1 &= \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \\1 &= \frac{1}{2} + \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{8}\right) \\1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \left(\frac{1}{16} + \frac{1}{16}\right)\end{aligned}$$

We can continue this process *ad infinitum*. Therefore,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 2$$

QED. Do you see this?

We now rearrange what we want to prove, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$, as follows:

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = e - 1$$

We now want to compare these two equations:

$$\begin{aligned}\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots &= e - 1 \\&\text{with} \\ \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots &= 2\end{aligned}$$

The first two terms are equal. We now compare the third and subsequent terms in the two infinite series:

$$\begin{aligned}\frac{1}{3!} &= \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6} < \frac{1}{4} \\ \frac{1}{4!} &= \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{24} < \frac{1}{8}\end{aligned}$$

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$$\frac{1}{5!} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{120} < \frac{1}{16}$$

ad infinitum ...

It is clear that the sum of the $(e - 1)$ series is less than 2. Therefore, the sum of the e series *must be less than 3*. When we add the terms in the e series, we get a value of $e = 2.7182818284 \dots$ QED.

Mathematicians make use of symbols to write a shorthand version of an infinite series. Let's now review these symbols.

- \sum represents the *sum* of all the terms in an infinite series (Note: Σ is the capital Greek letter *sigma*).
- \sum_1^∞ represents the sum of all the terms in an infinite series from $n = 1$ to $n \rightarrow$ (approaching) ∞ .
- n represents any positive integer.
- $n!$ is "n factorial", the product of all the integers from 1 to n or, as we have already noted, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$.

We can show that $0! = 1$. We note that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n - 1) \cdot n = n \cdot (n - 1) \cdot \dots \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Using this analysis, we observe:

$$\begin{aligned} 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5! = 720 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4! = 120 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 24 \\ 3! &= 3 \cdot 2 \cdot 1 = 3 \cdot 2! = 6 \\ 2! &= 2 \cdot 1 = 2 \cdot 1! = 2 \\ 1! &= 1 \cdot 0! = 1 \end{aligned}$$

Based upon the pattern indicated, we can conclude: Since $1 \cdot 0! = 1$, then $0! = 1$. QED.

Hence, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$. Using the above symbols and noting the beautiful pattern, we can now write a compact definition of e :

$$e = \sum_0^\infty \frac{1}{n!}$$

We can also algebraically play with these symbols and write e in a variety of ways:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$e = 1 + \sum_1^\infty \frac{1}{n!} \Rightarrow \sum_1^\infty \frac{1}{n!} = e - 1$$

$$e = 1 + 1 + \sum_1^\infty \frac{1}{(n+1)!} \Rightarrow \sum_1^\infty \frac{1}{(n+1)!} = e - 2$$

$$\text{Since } e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots, \text{ then } e = \sum_1^\infty \frac{1}{(n-1)!}$$

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The formula for compound interest is connected to e in a wondrous way. The expression for calculating compound interest is $\left(1 + \frac{1}{m}\right)^n$ where 1 is the capital, $\frac{1}{m}$ is the interest rate, and n is the number of years that the money is going to be deposited. Note, if the interest rate is 5%, then $m = 20$. From this simple formula, we can generate another one: $\left(1 + \frac{x}{n}\right)^n$, which, as $n \rightarrow \infty$ (equivalent to *continuous* compounding), is equal to e^x !

Using symbols, we get:

As $n \rightarrow \infty$, $\left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$ or, using limit notation from the calculus,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

When $x = 1$, we get $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$, our formula for e ! By replacing x with $(-x)$ and

observing the $(-x)$ raised to an odd power results in a negative number and $(-x)$ raised to an even power results in a positive number, we can derive the infinite series for e^{-x} :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \Rightarrow e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

$$\text{If } x = 1, \text{ then } e^{-1} = \frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \approx 0.3678794411\dots = \frac{1}{2.7182818284\dots}$$

We are finally ready to prove that e is irrational. To do this, we base our argument upon the infinite series represented by the reciprocal of e or $\frac{1}{e}$. Here is our starting point (noting that $1 = \frac{1}{0!}$):

$$\frac{1}{e} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

We consider what happens with the difference of consecutive partial sums of this series. We let $P(k)$ represent the k^{th} partial sum of this infinite series. By doing so, we immediately note the following:

$$P(k) - P(k-1) = \pm \frac{1}{k!}$$

Let's illustrate this truth for $k = 1, 2, 3, 4,$ and 5 :

k	$P(k)$	$P(k-1)$	$P(k) - P(k-1)$	$\pm \frac{1}{k!}$
1	$P(1) = 0$	$P(0) = 1$	$0 - 1 = -1$	$-\frac{1}{1!}$
2	$P(2) = \frac{1}{2}$	$P(1) = 0$	$\frac{1}{2} - 0 = \frac{1}{2}$	$+\frac{1}{2!}$
3	$P(3) = \frac{1}{3}$	$P(2) = \frac{1}{2}$	$\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$	$-\frac{1}{3!}$

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4	$P(4) = \frac{3}{8}$	$P(3) = \frac{1}{3}$	$\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$	$+\frac{1}{4!}$
5	$P(5) = \frac{11}{30}$	$P(4) = \frac{3}{8}$	$\frac{11}{30} - \frac{3}{8} = -\frac{1}{120}$	$-\frac{1}{5!}$

We now compare each pair of consecutive partial sums using common denominators. We note:

$$\begin{aligned}
 1) \quad & P(1) = \frac{0}{1} < \frac{1}{e} < \frac{1}{1} = P(0) \\
 2) \quad & P(1) = \frac{0}{2} < \frac{1}{e} < \frac{1}{2} = P(2) \\
 3) \quad & P(3) = \frac{2}{6} < \frac{1}{e} < \frac{3}{6} = P(2) \\
 4) \quad & P(3) = \frac{8}{24} < \frac{1}{e} < \frac{9}{24} = P(4) \\
 5) \quad & P(5) = \frac{44}{120} < \frac{1}{e} < \frac{45}{120} = P(4) \\
 6) \quad & P(5) = \frac{264}{720} < \frac{1}{e} < \frac{265}{720} = P(6) \\
 & \textit{ad infinitum} \dots
 \end{aligned}$$

The immediate necessity for concluding that $\frac{1}{e}$ (and e) is irrational should be evident from this demonstration. The following analysis explicates why this conclusion is justified.

Note the boundary conditions on $\frac{1}{e}$. Each successive boundary condition “boxes” $\frac{1}{e}$ between two rational numbers. Note also that each pair of bounding *numerators* differs by 1, and the denominators are factorials (2!, 3!, 4!, 5!, etc.) or $m!$

Let’s inspect these comparisons by looking at the third boundary condition. It tells us that if $\frac{1}{e}$ is *rational* its denominator *cannot* be a divisor of 6, because then it could be written $\frac{n}{6}$ for some integer n , and there is no such integer greater than 2 and less than 3.

Similarly, the fourth boundary condition proves that the denominator of $\frac{1}{e}$ cannot be a divisor of 24, and the fifth proves that it cannot be a divisor of 120, *ad infinitum*. It is clear that, by continuing with this logic, the *denominator* of $\frac{1}{e}$ cannot be a divisor of any $m!$ (for $m = 1, 2, 3, 4, \dots$). Note carefully that every integer k is a *divisor* of $m!$ (for all $m \geq k$). Since $\frac{1}{e}$ is bounded by these successful partial sums, the *numerator* of $\frac{1}{e}$ *cannot* be an integer. Hence, $\frac{1}{e}$ (and therefore e) *cannot* be a rational number.

QED.