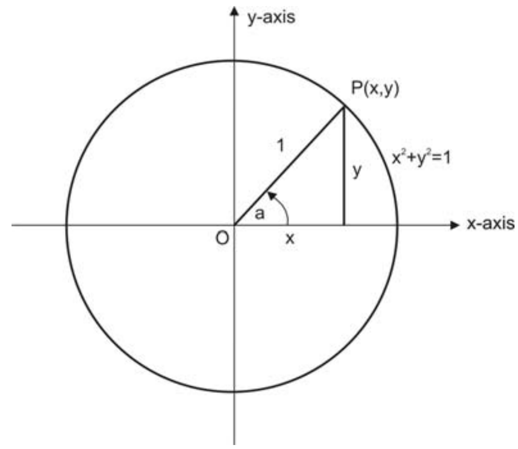
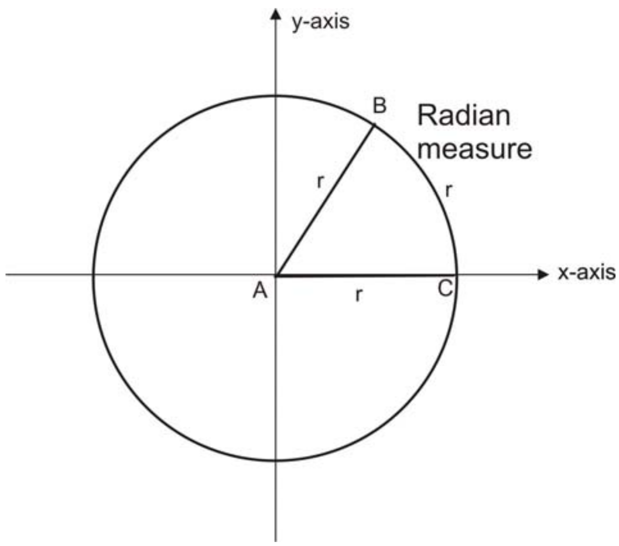
FROM THE RIGHT TRIANGLE TO WAVE MOTION

BY JAMES D. NICKEL

In studying the family of linear and quadratic equations we make frequent use of the Pythagorean Theorem. With linear equations, we calculate the slope as the ratio of the rise over the run in the context of a right triangle. We also derived a formula for calculating the distance between any two points in the Cartesian coordinate system by constructing a right triangle and calculating its hypotenuse. With quadratic equations, the use of a right triangle is critical in the formulation of an equation representing each of the four conic sections. Let's return to the circle to discover how it connects us to a new mathematical topic … trigonometry.

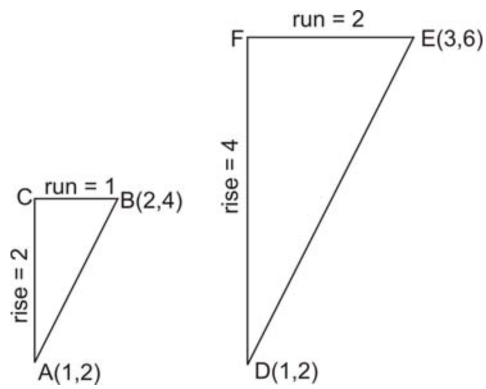


This circle is called the unit circle because its radius is 1 and its origin is at the coordinates (0, 0). By the Pythagorean Theorem, the equation of this circle is $x^2 + y^2 = 1$. If we let P(x,y) be a point on this circle and let the angle between the positive x-axis and the line OP (the hypotenuse) be *a*, then we can identify a new group of very important functions called *circular functions*. These functions are also called *trigonometric*[1] *functions*. These functions were first systematized by the Greek astronomer Hipparchus (ca. 180-125 BC), who lived in Rhodes and in Alexandria, to help solve astronomical problems.



The measure of arc BC can be in degrees or in radians (from radius). James Thomson, brother of the renowned physicist William Thomson or Lord Kelvin (1824-1907), first used this word in 1871. ***Radians*** (abbreviated rad) measure the length of the arc (a section of the perimeter or circumference of the circle) and it is the standard angular measure in the *International System of Units* (SI). One radian is the angle, measured at the center of a circle, that subtends (to extend underneath) an arc length of one radius along the circle's circumference. Since the circumference of a circle measures 2π radii ($c = 2\pi r$ where r = radius) and since each of these radii corresponds to a central angle of 1 radian, then we have $360° = 2\pi$ radians. Therefore, $180° = \pi$ radians, $90° = \pi/2$ radians, and $45° = \pi/4$ radians. $1° = 2\pi/360 = \pi/180 = 0.01745$ radians and 1 radian = 57.2958 degrees. Sometimes, in higher mathematics (namely the calculus), radian measure simplifies many formulas and that is why it is used.

[1] Trigonometry is Greek for "triangle measurement." The Greeks did not use this term to "name" their study of right triangles. As far as we know, the German mathematician Bartholomaeus Pitiscus (1561-1613) first used this term in 1595.

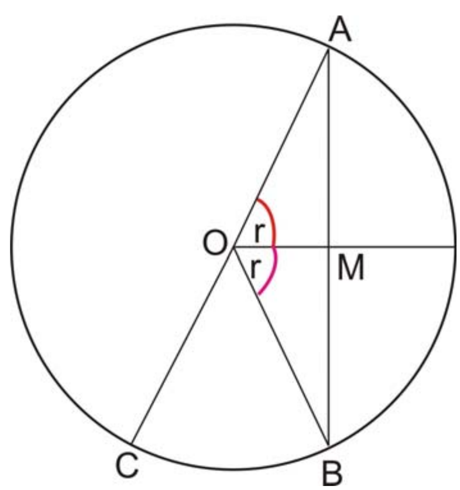


Builders and land surveyors in ancient Egypt noticed and made use of a pattern between two *similar* triangles. We can see this in our rise over run analysis. If the rise is 2 and the run is 1, then the slope of the line is 2. If the rise is 4 and the run is 2, then the slope is 4/2 = 2.

The two right triangles are similar. We note this symbolically by writing ΔABC ~ ΔDEF. In Geometry, we learn that the corresponding angles in two similar triangles are always *equal*. That is, ∠A = ∠D, ∠B = ∠E, and ∠C = ∠F (the two right angles). Although the corresponding sides of two similar triangles are not equal to each other (as in congruent triangles), they are *proportionate* to each other. By proportionate, we mean having the same or constant *ratio;* i.e., AC/DF = CB/FE = AB/DE. As a corollary, the ratio of adjacent sides in ΔABC will be equal to the corresponding ratio of adjacent sides in ΔDEF. For example, the rise over the run for ΔABC will be equal to the rise over the run in ΔDEF; i.e., AC/CB = 2/1 = 2 and DF/FE = 4/2 =2.

What is important to note is that this equivalent of ratios, i.e., AC/CB = DF/FE, will remain true for *every* right triangle having an acute angle equal to ∠A (*the unity in diversity principle).* This unity in diversity connection is the *key* to understanding the nature of trigonometric ratios.

The ancient Babylonians understood these trigonometric ratios in the context of the unit circle. The Greek astronomer Hipparchus borrowed from this source in the construction of his tables of ratios.



Hipparchus began with the circle and its circumference of 360°. He then divided the diameter ($\overline{AC}$) of the circle into 120 equal parts. Each part of the circumference and diameter was further divided into 60 parts (minutes) and each of these into 60 (seconds) more using the Babylonian sexagesimal fractions. Thus, for a given arc AB or $\overset{\frown}{AB}$ (in degrees), Hipparchus gave the number of units in the corresponding chord $\overline{AB}$ (remember, a chord is a line segment that joins two points of a curve) using the symbol *crd* α to represent the length of the chord of a *central angle* (an angle whose vertex is the center of a circle) whose measure is α.

The number of units in the chord $\overline{AB}$ corresponding to the number of degrees in $\overset{\frown}{AB}$ is equivalent to a well-known trigonometric *ratio*. In the figure, this ratio is AM/OA (the ratio of the length of the side opposite angle r to the length of the hypotenuse, the side opposite the right angle). Since Hipparchus worked with the diameter ($\overline{AC}$) instead of the radius ($\overline{OA}$), the ratio he worked with was AB/AC (which is proportionate to AM/OA. Why? Since ΔAMO and ΔABC are both right triangles, then ΔAMO ~ ΔABC). Note the measure of the central angle of the arc AB is 2r. Note also that the measure of this central angle and the measure of $\overset{\frown}{AB}$ ($m\overset{\frown}{AB}$) are equal. Hence, Hipparchus set the chord length; i.e., AB = crd 2r. Since AC = 120 units, we have this equation defining this ratio:

$$\frac{AM}{OA} = \frac{AB}{AC} = \frac{crd\ 2r}{120}$$

Based upon the work of Hipparchus, the Roman astronomer Claudius Ptolemy (ca. 85-ca. 165), who lived in Egypt, calculated a "table of chords", in 15 minute intervals, from 0° to 90°. These tables, the first formal presentation of trigonometry, are located in his famous work entitled *Almagest*.[2]

We now know this trigonometric ratio (AM/OA) to be the *sine* ratio. The etymology of the word "sine" has a fascinating history. We start our journey in India. The Hindu mathematician and astronomer Aryabhata the Elder (476-ca. 550) called it *ardha*-jya (meaning "half-chord"). It is $\overline{AM}$ in the figure. It was later abbreviated to *jya*. Note that *jya* could also mean "bow string." This makes sense since $\overset{\frown}{AB}$ resembles a bowstring. Now we travel to the Islamic culture. Arab translators turned this phonetically into *jiba* (without meaning in Arabic) and according to the Arabic practice of omitting the vowels in writing (similar to Hebrew), wrote it *jb*. Our journey now ends in Western Europe. European Arabic-to-Latin translators, having no knowledge of the Sanskrit (ancient Indic) origin, assumed *jb* to be an abbreviation of *jaib* (Arabic for "cove", "bay", "bulge", "bosom"). This also makes sense, since $\overset{\frown}{AB}$ looks like a curve or a bulge. When Gerard of Cremona (ca. 1114-1187) translated Ptolemy's *Almagest* in the 12th century, he translated *jaib* into the Latin equivalent *sinus* (from which we derive the English sine).

The mathematicians of the Christian west (i.e., Europe) named the trigonometric functions on the basis of a geometric interpretation by which they understood the *lengths* of specific line segments *related to the central angle* of the unit circle. Let's see how they did it (walk slowly through this explanation; make sure that you understand and visually note each inference by referring to the figure).

> We turned aside, not indeed to the uplands of the Delectable Mountains, but into a strange corridor of things like anagrams and acrostics called Sines, Cosines and Tangents ... I have never met these creatures since. With my third and successful examination, they passed away like the phantasmagoria of a fevered dream. I am assured that they are most helpful in engineering, astronomy and things like that. It is very important to build bridges and canals and to comprehend all the stresses and potentialities of matter, to say nothing of counting all the stars and even universes and measuring how far off they are, and foretelling eclipses, the arrival of comets and such like. I am very glad there are quite a number of people born with a gift and a liking for all of this; like great chess-players who play sixteen games at once blindfold and die quite soon of epilepsy. Serve them right! I hope the Mathematicians, however, are well rewarded. I promise never to backleg their profession nor take the bread out of their mouths.
>
> Winston Churchill, "Examinations," *My Early Life (1874-1904), p. 26.*

[2] Arabic for "the greatest." This work was originally entitled *Syntaxis mathematica* (mathematical collection) in which Ptolemy also enshrined the geocentric cosmology. Arab scientists renamed this work after translating it in 827.

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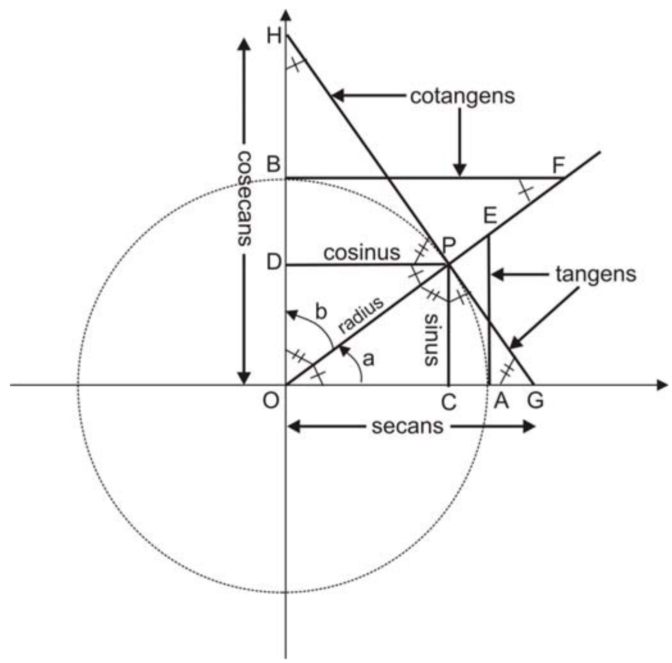
The radius of the unit circle, OP = 1, can rotate in a counterclockwise direction. Let's assume the initial position of the radius to be $\overline{OA}$. We then rotate the radius segment to $\overline{OP}$, its terminal position. As we do so, the radius segment will generate an angle equal to *a*.

The tangents to the circle at points A and B meet the extension of the radius $\overline{OP}$ at E and F respectively. The line segments, $\overline{PC}$ (which is perpendicular to $\overline{OA}$) and $\overline{PD}$ (which is perpendicular to $\overline{OB}$) form three right triangles, ΔOCP, ΔOAE, and ΔOBF, which are similar. Note also all the other right triangles in the figure (e.g., ΔHOG, ΔODP, ΔHDP, ΔPCG, ΔOPG). Based upon similarity relationships between right triangles, note the angles marked with one tick (′) are equal and the angles marked with two ticks (″) are also equal.



Consider the line segment $\overline{HG}$ tangent to the circle at P. It can be divided into two segments, PG and PH such that: (1) PG = AE, and (2) PH = BF. Why? ΔOAE ≅ ΔOPG and ΔHPO ≅ ΔOBF.

Note that OA = OB = OP = 1. Since ∠a + ∠b = 90° (the measure of a right angle), we call ∡b the *complementary*[3] angle of ∡a (Remember, two angles are complimentary if their sum is 90°). The *co-* prefix in cosinus, cosecant, and cotangent carry the complementary angle concept in their etymology.

From these definitions and observations, we can develop the following ratios and equalities:

radius = OA = OB = OP = 1

sinus a = PC/OP = PC = OD (technically, *sinus* a is the *projection*[4], $\overline{PC}$, of $\overline{OP}$ on the y-axis).

cosinus a = OC/OP = OC = PD (*cosinus* a is the *projection*, $\overline{OC}$, of $\overline{OP}$ on the x-axis).

tangens a = AE/OA = AE = PG = PC/OC (*tangens* a is the *projection*, $\overline{AE}$, of $\overline{OP}$ on the line tangent to the circle at A).

cotangus a = BF/OB = BF = PH = OC/PC (*cotangens* a is the *projection*, $\overline{BF}$, of $\overline{OP}$ on the line tangent to the circle at B).

secans a = OE/OA = OE = OG = OP/OC (*secans* a is the *projection*, $\overline{OG}$, of $\overline{OP}$ on the x-axis external to circle O).

cosecans a = OF/OB = OF = OH = OP/PC (*cosecans* a is the *projection*, $\overline{OH}$, of $\overline{OP}$ on the y-axis external to circle O).

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[3] *Complement* comes from the Latin meaning "to fill out" or "to complete."

[4] Think of projections as *shadows*. There is a more technical definition but a shadow gives you the idea.

In 1583, Thomas Fincke (1561-1656) introduced the other functions that had been known by other names. Tangent (Latin: *tangere*) means "to touch." Secant (Latin: *secare*) means "to cut." The English mathematician and astronomer Edmund Gunter (1581-1626) first suggested cosine, cotangent, and cosecant in 1620 (replacing *sinus complementi* and *tangens complementi:* the *sinus* and *tangens* of the complimentary angles).

Today, we abbreviate sine as *sin*, cosine as *cos*, tangent as *tan*, cotangent as *cot*, secant as *sec*, and cosecant as *csc*. We can summarize all the right triangle ratios and their corresponding names as follows (see if you can make sense of these ratios given the above discussion):

sin A = BC/AB (side **o**pposite ∡A over the **h**ypotenuse). Memory aid: *soh*.

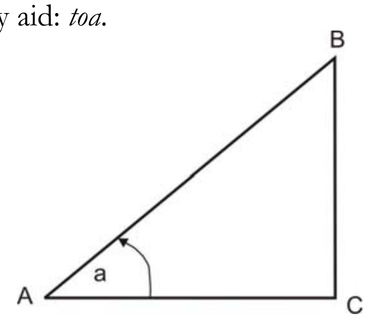
cos A = AC/AB (side **a**djacent ∡A over the **h**ypotenuse). Memory aid: *cah*.

tan A = BC/AC (side **o**pposite ∡A over side **a**djacent ∡A). Memory aid: *toa*.

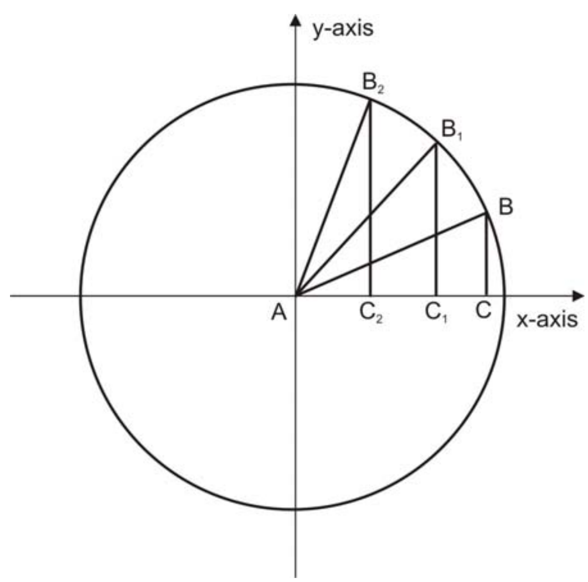
cot A = AC/BC (side adjacent ∡A over side opposite ∡A).

sec A = AB/AC (hypotenuse over side adjacent ∡A).

csc A = AB/BC (hypotenuse over side opposite ∡A).



The *sin*, *cos*, and *tan* ratios are most commonly used (scientific calculators have keys labeled as such). Let's investigate the nature of these functions as ∡A varies in measurement from 0° to 90°. First, consider the unit circle (hypotenuse = 1). Since sin A = opposite/hypotenuse, then sin A = BC/1 = BC. cos A = AC/1 and tan A = BC/AC.

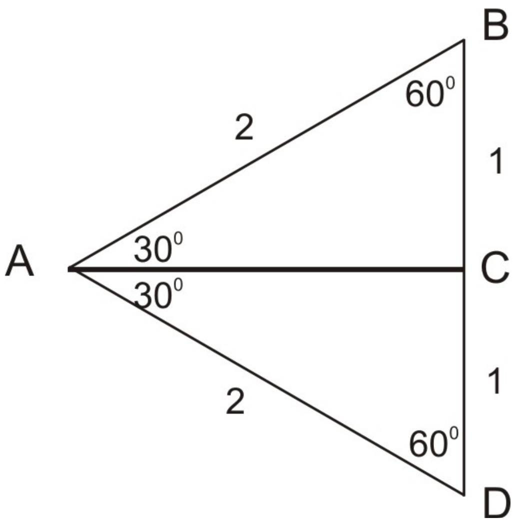


As ∡A gets smaller, then BC gets smaller and AC gets larger. In fact, if ∠A = 0, then sin A = 0 since BC = 0. cos A = 1 since AC = 1. tan A = BC/AC = 0.

As ∡A gets larger, then BC gets larger and AC gets smaller. If ∠A = 90°, then sin A = 1 since BC = 1. cos A = 0 since AC = 0. tan A = BC/AC = indeterminate since we cannot divide by 0. We now know the behavior of three trigonometric functions as ∡A varies in measurement from 0° to 90°:

A	sin A	cos A	tan A
0°	0	1	0
90°	1	0	—

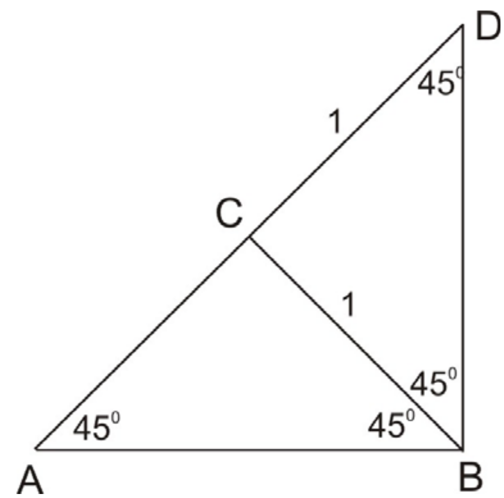
What happens in between 0° to 90°? Let's consider at the equilateral ΔABD with each side of length 2. This type of triangle is also equiangular; i.e., each angle equals 60°.



Construct an *angle bisector*[5] $\overline{AC}$ of ∠BAD. Therefore, ∠BAC = ∠DAC = 30°. Since AB = AD = BD = 2, then CB = CD = 1 because the angle bisector of ∠BAD is also the *perpendicular bisector*[6] of $\overline{BD}$. What is the length of $\overline{AC}$ or AC? By the Pythagorean theorem, $AD^2 = AC^2 + CD^2$. Hence, $AC^2 = AD^2 - CD^2$. Therefore, AC is calculated as follows:

$$AC = \sqrt{AD^2 - CD^2} = \sqrt{4-1} = \sqrt{3}$$

Knowing these lengths, we can determine the trigonometric ratios for 30° and 60°. sin 30° = ½, cos 30° = $\frac{\sqrt{3}}{2}$, and tan 30° = $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.[7] sin 60° = $\frac{\sqrt{3}}{2}$, cos 60° = $\frac{1}{2}$, and tan 60° = $\sqrt{3}$.



Now, let's consider 45°. First, draw two line segments $\overline{AB}$ and $\overline{DB}$ perpendicular to each other (∠ABD = 90°). Second, draw an angle bisector $\overline{BC}$ of ∠ABD with length of 1 unit. Construct a $\overline{AD}$ perpendicular to $\overline{BC}$ of 2 units in length so that $\overline{BC}$ is the perpendicular bisector of $\overline{AD}$. ΔBCD is now a right triangle with legs DC = BC = 1. Hence, DB, the length of the hypotenuse of ΔBCD, is calculated as follows:

$$DB = \sqrt{DC^2 - BC^2} = \sqrt{1+1} = \sqrt{2}$$

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[5] Remember, an angle bisector divides a given angle in half.

[6] Remember, the perpendicular bisector is a line divides a given line segment in half intersecting it at a 90° angle.

[7] It is custom in mathematics to not leave a root in the denominator of a fraction; i.e., $\sqrt{3}$ in $\frac{1}{\sqrt{3}}$. In this case, we *rationalize the denominator* by multiplying the numerator and denominator by $\sqrt{3}$. Hence, $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Knowing these three lengths, we can determine the trigonometric ratios for 45°. sin 45° = cos 45° = $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (after rationalizing the denominator), and tan 45° = 1.

Our table now looks as follows:

A(°)	A (rad)	sin A	cos A	tan A
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{\sqrt{3}}{3} = 0.577$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} = 0.707$	$\frac{\sqrt{2}}{2} = 0.707$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{1}{2} = 0.5$	$\sqrt{3} = 1.732$
90°	$\frac{\pi}{2}$	1	0	—

Mathematicians use a variety of methods to calculate the ratios in between these angles. For example, without going into any derivation (it requires methods of the calculus), both *sin* and *cos* can be approximated through the use of *power series*. A power series in *x*, where the *a's* are constants, is defined as follows:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

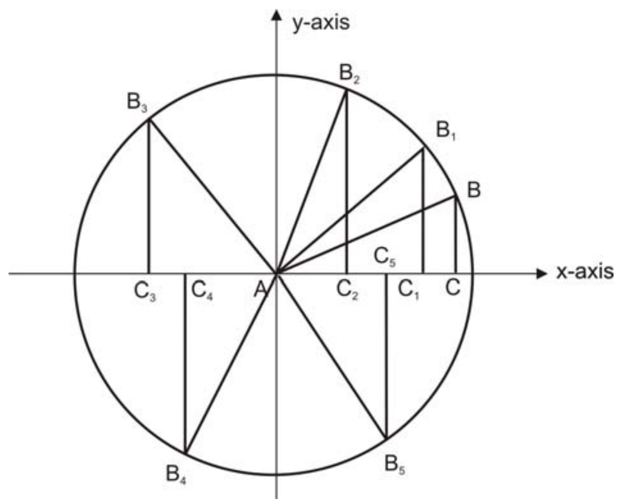
If *x* is measured in radians, then both *sin x* and *cos x* can be approximated in terms of a power series as follows (NB. n! means "n factorial":

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots,$$

The Swiss mathematician Leonhard Euler (1707-1783) used these definitions of sin x and cos x to derive a *truly amazing* formula (called *Euler's Identity*); i.e., $e^{i\pi} + 1 = 0$.

Older mathematics textbooks had "trig tables" in the back of the book. With the advent of the scientific calculator, we no longer need these tables. You can enter any degree or radian on the keys of these calculators and then punch the *sin*, *cos*, or *tan* key to get the corresponding approximate ratio in decimal form.

How do these ratios behave between 90° to 360°? For Quadrant I (from 0° to 90°), all ratios are positive. Let's consider an angle, $\measuredangle B_3AC$, in Quadrant II (from 90° to 180°). This angle is an obtuse angle and the slope of line segment $\overline{B_3A}$ is negative. Why? The coordinates of A are (0, 0) and the coordinates of B_3 are, in general, (-x, y). The slope is $\frac{\text{rise}}{\text{run}} = \frac{y}{-x} = -\frac{y}{x}$. We determine the trigonometric ratios for any obtuse angle in Quadrant II by applying our definitions to $\measuredangle B_3AC_3$ (remember, the radius of this unit circle is 1):



$$\sin \angle B_3AC_3 = y$$
$$\cos \angle B_3AC_3 = -x$$
$$\tan \angle B_3AC_3 = -\frac{y}{x}$$

Let's consider an angle, $\measuredangle B_4AC$, in Quadrant III (from 180° to 270°). This angle is also an obtuse angle and the slope of line segment $\overline{B_4A}$ is positive because the coordinates of A are (0, 0) and the coordinates of B_4 are, in general, (-x, -y). The slope is $\frac{\text{rise}}{\text{run}} = \frac{-y}{-x} = \frac{y}{x}$. We determine the trigonometric ratios for any obtuse angle in Quadrant III by applying our definitions to $\measuredangle B_4AC_4$:

$$\sin \angle B_4AC_4 = -y$$
$$\cos \angle B_4AC_4 = -x$$
$$\tan \angle B_4AC_4 = \frac{-y}{-x} = \frac{y}{x}$$

Finally, let's consider an angle, $\measuredangle B_5AC$, in Quadrant IV (from 270° to 360°). This angle is also an obtuse angle and the slope of line segment $\overline{B_5A}$ is negative because the coordinates of A are (0, 0) and the coordinates of B_5 are, in general, (x, -y). The slope is $\frac{\text{rise}}{\text{run}} = \frac{-y}{x} = -\frac{y}{x}$. We determine the trigonometric ratios for any obtuse angle in Quadrant IV by applying our definitions to $\measuredangle B_5AC_5$:

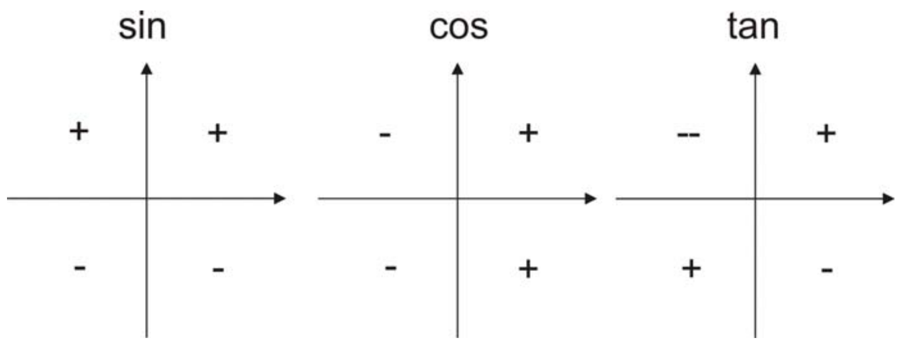
$$\sin \angle B_5AC_5 = -y$$
$$\cos \angle B_5AC_5 = x$$
$$\tan \angle B_5AC_5 = -\frac{y}{x}$$

From this information, we can complete our table of trigonometric ratios for select angles between 0° to 360°:

A(°)	A (rad)	sin A	cos A	tan A
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{\sqrt{3}}{3} \approx 0.577$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$	$\frac{\sqrt{2}}{2} \approx 0.707$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{2} = 0.5$	$\sqrt{3} \approx 1.732$
90°	$\frac{\pi}{2}$	1	0	—
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$	$-\frac{1}{2} = -0.5$	$-\sqrt{3} \approx -1.732$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$	$-\frac{\sqrt{2}}{2} \approx -0.707$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2} = 0.5$	$-\frac{\sqrt{3}}{2} \approx -0.866$	$-\frac{\sqrt{3}}{3} \approx -0.577$
180°	π	0	-1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2} = -0.5$	$-\frac{\sqrt{3}}{2} \approx -0.866$	$\frac{\sqrt{3}}{3} \approx 0.577$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$	$-\frac{\sqrt{2}}{2} \approx -0.707$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$	$-\frac{1}{2} = -0.5$	$\sqrt{3} \approx 1.732$
270°	$\frac{3\pi}{2}$	-1	0	—
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$	$\frac{1}{2} \approx 0.5$	$-\sqrt{3} \approx -1.732$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$	$\frac{\sqrt{2}}{2} \approx 0.707$	-1

A(°)	A (rad)	sin A	cos A	tan A
330°	$\frac{11\pi}{6}$	$-\frac{1}{2} \approx -0.5$	$\frac{\sqrt{3}}{2} \approx 0.866$	$-\frac{\sqrt{3}}{3} \approx -0.577$
360°	2π	0	1	0

From this table we note this rule: as an angle increases through the four quadrants, the trigonometric ratios remain the same except for the sign. The sign changes per Quadrant as follows:



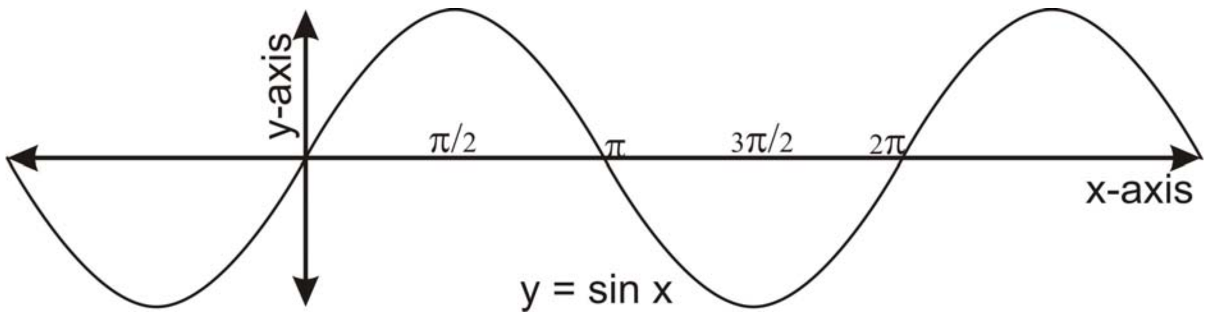
Note that for angles greater than 360°, the ratio sequence starts over. That is, sin 390° = sin 30°. Note also that we have been measuring our angles in a counterclockwise direction starting in the "three o'clock position" on the coordinate grid. This method of measuring angles is customary. We can measure these angles in a clockwise direction (as negative angles) and we note these identities:

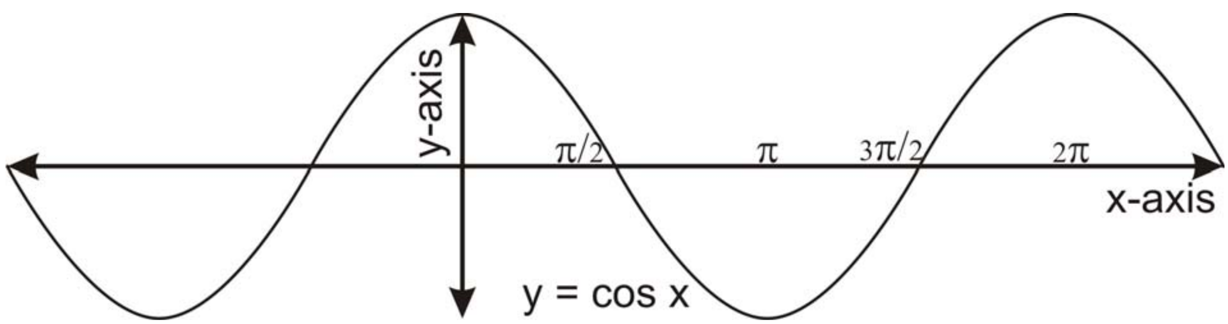
sin (-a) = sin (360 – a) = -sin a
cos (-a) = cos (360 – a) = cos a
tan (-a) = tan (360 – a) = -tan a

Try these identities with a few examples:

sin (-45) = sin (315) = -sin 45 = -0.707
cos (-90) = cos (270) = cos 90 = 0
tan (-120) = tan (240) = -tan (120) = 1.732

If we graph the equations y = sin x and y = cos x (where x is in radians), we can see the cyclical or periodic pattern of these functions. We can change the human voice and musical sounds into electrical current through the medium of a microphone. If we connect this microphone to a special instrument, called an oscilloscope, you will see a graph that looks similar to the ones pictured below.





The graphs of sounds that are pleasing to the ear reflect an order and regularity. We can describe this periodicity as a function:

$$y = f(t) = D \sin 2(ft)$$

By definition, y represents the *displacement*, *D* the *amplitude*, *f* the *frequency* and *t* the *time*. This equation describes very simple sounds, like those of a tuning fork. For complex sounds, the equation still holds, but mathematicians expand it as the summation of a mathematical series, called the Fourier series after its originator, Joseph Fourier (1768-1830). The work of this Frenchman in the analysis of heat provided the foundation for the mathematical study of music, a study that has enabled man in the 20th century to construct musical electronic gadgets like the synthesizer.[8]

Using the Fourier series, we can describe *middle C* on the piano as follows:

$$y = f(t) = \sin 2(512t) + 0.2 \sin 2(1024t) + 0.25 \sin 2(1536t) + 0.1 \sin 2(2048t) + 0.1 \sin 2(2560t)$$

Using mathematics, we can describe every sound that is pleasing to the ear as sinusoidal functions. The order and harmony of true music will create order and harmony in those who listen to it and play it. Those sounds that are not pleasing to the ear we call noise. We cannot describe noise and dissonant

[8] Joseph Fourier, *The Analytical Theory of Heat*, trans. Alexander Freeman (Cambridge: Cambridge University Press, 1878).

music in terms of the mathematics above. The disorder of noise, and much of the popular "rock music" of today could be proved mathematically to be noise and will create disorder in those who listen to it and play it.

Any wavelike, or regular, motion that models musical vibrations (e.g., the path of meandering rivers, ocean tides, the crest and trough of ocean waves, the ebb and flow of alternating electric current, and the majestic rotation of galaxies) can be also described mathematically in terms of trigonometric functions.

In conclusion, note the wondrous order and complexity in music! Comprehending it involves a thorough knowledge of trigonometry *and the foundation of trigonometry is the simple right triangle.* Note also the progression from a simple right triangle to the complex order of music. In fact, sinusoidal functions not only perfectly describe sound waves, but they also describe the distinct, wavelike motion of visible light and in fact, *the entire electromagnetic spectrum*. What *unity in diversity!* Who is the originator of light and the electromagnetic spectrum? The architect is the Biblical God, the ultimate and eternal One and the Many.