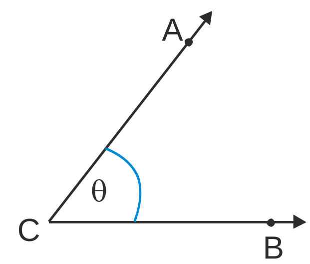
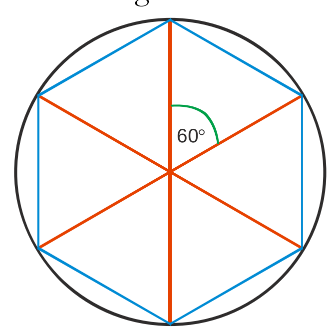
Why Radians?

By James D. Nickel



This essay is a brief summary of the mathematics of angle measurement.[1] The Greek geometer Euclid (ca. 300 BC) defined a plane angle (two dimensions) as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." A little esoteric, isn't it?

Using modern terminology, we first define a ray as "half of a line; it has one endpoint with the other end extending to infinity." In the figure, we see two rays: $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Next, an angle is defined as "the union of two rays with a common endpoint." In the figure, the common endpoint is C. What does the word "union" mean? To answer this, we must differentiate between identification and measurement. For example, in the figure, we identify the line segment between points C and A as $\overline{CA}$ or $\overline{AC}$. The distance between these two points is a measurement (in geometry, this distance is symbolized as CA or AC). With angles, we identify them in a variety of ways. We can identify the angle in the figure as ∠C, ∠ACB, ∠BCA, or θ (The Greek letter "theta"). To measure this angle, we must measure its "sweep" from $\overrightarrow{CB}$ to $\overrightarrow{CA}$ (counterclockwise), or from $\overrightarrow{CA}$ to $\overrightarrow{CB}$ (clockwise). Hence, union could mean identification or measurement depending on what we are after. In what way do we *measure* angles?



The ancient Babylonians divided the circle into 360 equal parts to help them measure angles. Historians are unable to establish with certainty the ultimate reasons for this but it is likely related to their use of the sexagesimal (base 60) system. Some have conjectured the reason to be that 360 is near the number of days in one year. Others have noted that a circle divides naturally into six equal parts, each subtending a chord[2] equal to the measure of the circle's radius.

The figure at right shows six equilateral triangles formed by a regular hexagon inscribed in a circle. Each angle of the triangle consists of 60 parts of 360. One of these parts is called a degree (the symbol is °).[3] Each degree can be further divided into sixty parts (called minutes[4]: the symbol is ′) and each minute can be divided into sixty parts (called seconds[5]: the symbol is ″). With precision instruments, we can measure degrees, in base 60, to fractions of a second; e.g., 60°14′15.25″.

The scholars of the French revolution (late 18th century) tried to decimalize angle measurement but it was not a success.[6] A remnant of this thrust is contained in the *grad* measure (based upon a circle divided into 400

[1] The source for much of the information in this essay is the excellent book by Eli Maor, *Trigonometric Delights* (Princeton: Princeton University Press, 1998).

[2] A chord is a line segment that connects two points on a circle. To subtend means to "extend under." A chord is a function of the angle it subtends at the center.

[3] The ancient Greeks used μοιρα (*moira*) to represent one part of 360. Islamic scholars translated this word into *daraja* and this word became *de gradus* in Latin. The journey to the English *degrees* is one simple step from the Latin.

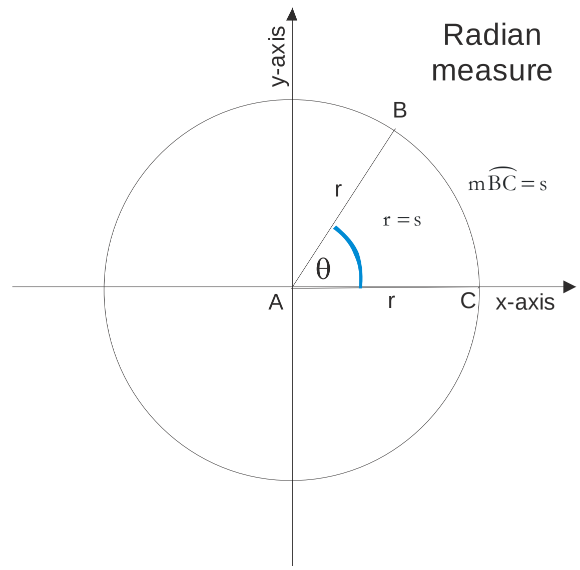
[4] The Ancient Greeks called a sixtieth part of a degree the "first part." Latin scholars translated this as *pars minuta prima* (first small part). Hence, the *minute.*

[5] The Ancient Greeks called a sixtieth part of the first part the "second part." Latin scholars translated this as *pars minuta secunda* (second small part). Hence, the *second.*

[6] They also tried a ten-day week for a while but that idea failed too.

equal parts). One grad is therefore equal to $\frac{1}{400}$ of a circle. Hence, 90°, the measure of a right angle, is 100 grads.

The latest unit of angular measure is the *radian* or *circular measure*. James Thomson, brother of the renowned physicist William Thomson or Lord Kelvin (1824-1907), first used this word in 1871 (it is derived from radius). Radians (abbreviated rad) measure the *length of a circular arc* (a part of the perimeter or circumference of the circle) and it is the standard angular measure in the *International System of Units* (SI). In the figure, the measure of arc BC (symbol: $m\overset{\frown}{BC}$)[7] can be in degrees or in radians. If in radians, the length of that arc is the measure of ∠BAC; i.e., ∠BAC = θ.



One radian is the angle, measured at the center of a circle, that subtends an arc length of *one radius* along the circle's circumference. In the figure, θ = 1 rad. *r* is the radius of the circle and r = 1. Hence, $r = m\overset{\frown}{BC}$ and if $m\overset{\frown}{BC} = s$, then r = s.

Since the circumference of a circle measures 2π radii (c = 2πr) and since each of these radii corresponds to a central angle of 1 radian (where r = 1), then 360° = 2π rad. Therefore, 180° = π rad, 90° = $\frac{\pi}{2}$ rad, and 45° = $\frac{\pi}{4}$ rad. 1° = $\frac{2\pi}{360} = \frac{\pi}{180}$ ≈ 0.01745 rad and 1 rad ≈ 57.2958°.

Why use radians? The primary reason is that it simplifies many formulas. For example, from geometry we know this proportion: the ratio of the circumference of a circle (C = 2πr) to 2π rad is the same as the ratio of the arc length *s* to θ (in rad). In symbols, $\frac{2\pi r}{2\pi} = \frac{s}{\theta} \Rightarrow r = \frac{s}{\theta} \Rightarrow s = r\theta$.[8] If θ is in degrees, this formula would not be as compact or as beautiful in its simplicity; it would be $s = \frac{\pi r\theta}{180}$ where $\frac{\pi}{180}$ is the extra factor.

Similarly, the ratio of the area of a circle ($A = \pi r^2$) to 2π rad is the same as the ratio of the area of a circular sector[9] to θ (in rad). We let A = the area of this sector. In symbols, $\frac{\pi r^2}{2\pi} = \frac{A}{\theta} \Rightarrow \frac{r^2}{2} = \frac{A}{\theta} \Rightarrow A = \frac{r^2\theta}{2}$. If θ is in degrees, this formula becomes $A = \frac{\pi r^2\theta}{360}$ (again, $\frac{\pi}{180}$ is the extra factor).

We also note that, for a small angle θ (in rad), sin θ = θ. Note, sin 1° = 0.174524064. When we convert 1° to radians, we get 1° = $\frac{\pi}{180}$ ≈ 0.174532925. We have agreement to the ten-thousandths place. When using the

---

[7] $\overset{\frown}{BC}$ identifies the arc on the circle from point B to point C.
[8] If r = 1 (the unit circle), then s = r.
[9] In a circle, a sector is a region bounded by two radii of the circle and by the arc of the circle whose endpoints lie on those radii.

limit notation of the calculus, we get: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$. Because of this and other reasons, radian measure engenders efficiency and beauty in the calculus and that is why it is used.